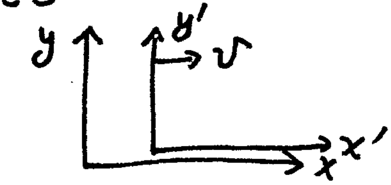
Lect 17 Relativistic mechanics



☆ Addition of velocity:

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \Rightarrow \frac{dx'}{c\,dt'} = \frac{-\gamma\beta c\,dt + \gamma dx}{\gamma c\,dt - \gamma\beta dx} = \frac{-\beta + \frac{dx}{c\,dt}}{1 - \beta\frac{dx}{dt}}$$

or $$\boxed{u_x' = \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2}}$$

$$\frac{dy'}{c\,dt'} = \frac{dy}{\gamma c\,dt - \gamma\beta dx} = \frac{1}{\gamma}\,\frac{dy/c\,dt}{1 - \beta\frac{dx}{c\,dt}}$$

or $$\boxed{u_y' = \frac{dy'}{dt'} = \frac{1}{\gamma}\,\frac{u_y}{1 - u_x v/c^2}}$$

similarly $$\boxed{u_z' = \frac{1}{\gamma}\,\frac{u_z}{1 - u_x v/c^2}}$$

The inverse transformation: change the sign of v

$$\boxed{u_x = \frac{u_x' + v}{1 + u_x' v/c^2}, \quad u_y = \frac{1}{\gamma}\,\frac{u_y'}{1 + u_x' v/c^2}, \quad u_z = \frac{1}{\gamma}\,\frac{u_z'}{1 + u_x' v/c^2}}$$

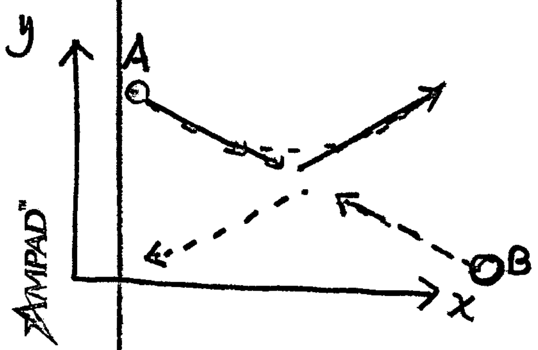
if $\begin{cases} u_x = c \\ u_y = u_z = 0 \end{cases} \Rightarrow \begin{cases} u_x' = \dfrac{c - v}{1 - v/c} = c \\ u_y' = u_z' = 0 \end{cases}$

if $\begin{cases} u_x = 0 = u_z \\ u_y = c \end{cases} \Rightarrow \begin{cases} u_x' = -v \\ u_y' = \dfrac{c}{\gamma} \\ u_z' = 0 \end{cases} \Rightarrow \sqrt{u_x'^2 + u_y'^2} = v^2 + c^2(1 - v^2/c^2) = c^2$

light velocity does not change.

§ Relativistic momentum and energy

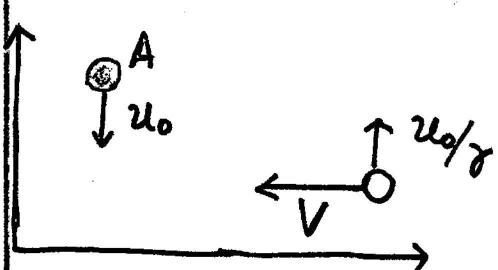
Let us consider the collision between two identical ball A and B



in the lab frame, the trajectory of A and B are symmetric as show on the left. Their velocities are opposite in direction and the same in magnitude both before and after the collision. (Their velocities along the x-direction do not change.)

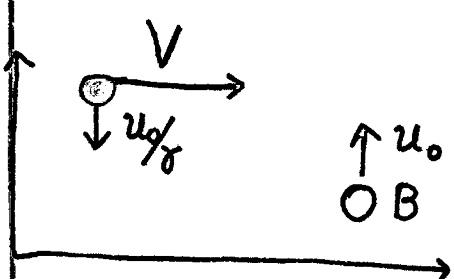
Now let us change frame to A's frame, which move with A along the x-axis. In this frame, before the collision, A's velocity is only along the -y-direction, we denote it as u_0.

A's frame



We also conside B's frame, which co-moves with B along $-x$ axis. In this frame, B's velocity is along $\hat{y}$-axis. Due to the symmetry, its velocity is also u_0.

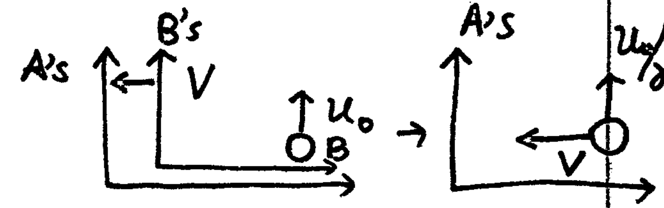
B's frame



Denote the relative velocity between A's and B's frame as V.

before the collision

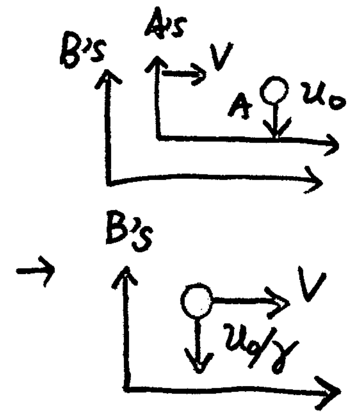
Let's calculate the velocity of B in A's frame.



$$\therefore \; u_{B,x}^{(A)} = \frac{0 - V}{1 - 0\cdot V/c^2} = -V$$

$$u_{B,y}^{(A)} = \frac{1}{\gamma}\,\frac{u_0}{1 - 0\cdot V/c^2} = \frac{u_0}{\gamma}$$

Similarly, we also have the velocity of A in B's frame



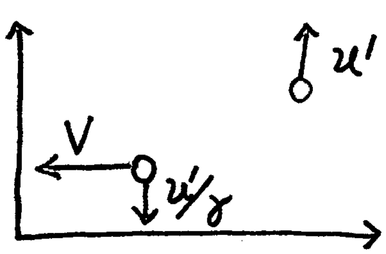
$$u_{A,x}^{(B)} = \frac{0 + V}{1 + 0\cdot V/c^2} = V$$

$$u_{A,y}^{(B)} = \frac{-u_0}{\gamma(1 + 0\cdot V/c^2)} = -\frac{u_0}{\gamma}$$

Now let us consider the situation after the collision.

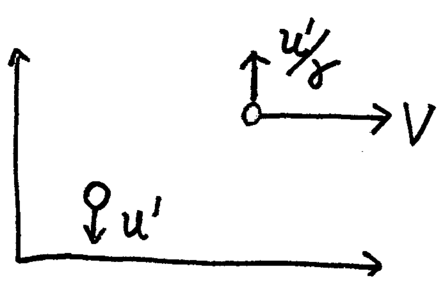
A and B's velocities doesn't change along x-direction, but changes direction of their y-components.

A's frame



By similar reasoning, we have their relations as plotted on the left.

B's frame



After the collision

we are looking for a new definition of momentum $\vec{P} = m(w)\vec{W}$ where $m(w)$ only depends on the magnitude of W. $\vec{P}_{tot}$ should be conserved during the collision.

In A's frame, the x-direction momentum completely comes from B. It speed $w = (V^2 + \frac{u_0^2}{\gamma^2})^{1/2}$ (before) and $w' = (V^2 + \frac{u_0'^2}{\gamma^2})^{1/2}$ (after)

Then from

$$m(w)\, V = m(w')\, V \Rightarrow w = w' \Rightarrow u' = u.$$

we assume $m(w)$ should be a monotonic function

Then we check the y-direction momentum conservation

$$-m(u_0)\, u_0 + m(w)\frac{u_0}{\gamma} = m(u_0)\, u_0 - m(w)\frac{u_0}{\gamma}$$

$$\Rightarrow \quad m(w) = \gamma\, m(u_0)$$

let us consider the limit $u_0 \to 0$, then $m(u_0) = m_0 \leftarrow$ rest mass

$w = V$

$$\Rightarrow \quad m(V) = \gamma\, m_0 = \frac{m_0}{\sqrt{1 - V^2/c^2}}.$$

Thus $\vec{P} = \dfrac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}$ for a mass point with rest mass m_0 and velocity $\vec{u}$.

Let us also check $m(\omega) = \frac{1}{\sqrt{1 - V^2/c^2}} m(u_0)$

$$\omega = \left(V^2 + \frac{u_0^2}{\gamma^2}\right)^{1/2} \Rightarrow \frac{1}{\sqrt{1-\omega^2/c^2}} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2} - \frac{u_0^2}{\gamma^2 c^2}}}$$

$$\gamma^{-2} = 1 - \frac{V^2}{c^2}$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}\sqrt{1-\frac{u_0^2}{c^2}}}$$

$$\Rightarrow m(\omega) = \frac{m_0}{\sqrt{1-\omega^2/c^2}} = \frac{m_0}{\sqrt{1-V^2/c^2}\sqrt{1-\frac{u_0^2}{c^2}}} = \frac{1}{\sqrt{1-V^2/c^2}} m(u_0)$$

§ energy

$$K_b - K_a = \int_a^b \frac{d\vec{p}}{dt} \cdot d\vec{r} = \int_a^b \frac{d}{dt}\left[\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right] \cdot \vec{u}\, dt$$

$$= \int_a^b \vec{u} \cdot d\left[\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right]$$

$$= \frac{m_0 u^2}{\sqrt{1-u^2/c^2}}\Bigg|_{u_a}^{u_b} - \int_a^b \frac{m_0 u\, du}{\sqrt{1-u^2/c^2}}$$

$$\vec{u} \cdot d\vec{u} = u\,du$$

$$= \frac{m_0 u^2}{\sqrt{1-u^2/c^2}}\Bigg|_{u_a}^{u_b} + m_0 c^2 \sqrt{1-u^2/c^2}\,\Bigg|_{u_a}^{u_b}$$

We choose the 'a' point at which the point is at rest, i.e. $u_a = 0$

$$\Rightarrow K_b = \frac{m_0 u_b^2}{\sqrt{1-u_b^2/c^2}} + m_0 c^2 \sqrt{1-u_b^2/c^2} - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-u_b^2/c^2}} - m_0 c^2$$

$$\frac{m_0 c^2}{\sqrt{1-u^2/c^2}} = mc^2 \leftarrow \text{Einstein's mass - energy relation}$$

$$m_0 c^2 \swarrow \text{rest energy}$$

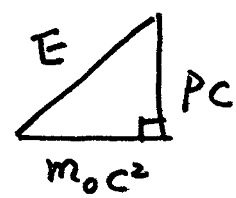
$$K = mc^2 - m_0 c^2 . \quad \text{As } u \to 0 . \ \frac{1}{\sqrt{1-u^2/c^2}} \approx 1 + \frac{u^2}{2c^2}$$

$$\Rightarrow K \approx \frac{1}{2} m u^2 . \leftarrow \text{Newtinian kinetic energy.}$$

AMPAD

define $E = \dfrac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}$ $\Rightarrow E^2 = \dfrac{m_0^2 c^4}{1-\frac{u^2}{c^2}} = m_0^2 c^4 + \dfrac{m_0^2 u^2 c^2}{1-\frac{u^2}{c^2}}$

$$= m_0^2 c^4 + p^2 c^2$$



☆ we can also think massless particles

$m_0 = 0$

$\Rightarrow E^2 = (pc)^2$ or $E = pc$.

The $p = \lim_{m_0 \to 0} \dfrac{m_0 u}{\sqrt{1-u^2/c^2}}$ $\Rightarrow$ u must be equal to c, in order to have nonzero momentum.

photon, graviton.

Example: ① π^0 ($m_0 = 2.4\times10^{-28}$ Kg), $v = 0.8c$, what's its Kinetic energy?

$K = (\gamma-1)m_0c^2$, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}} = 1.67 \Rightarrow K = 0.67\,m_0c^2 \approx 1.4\times10^{-11}$ J

② momentum of an electron with speed $v_1 = 4\times10^7$ m/s, $v_2 = 0.98c$

$P_1 = \gamma m v = 1.01\,mv$, $P_2 = \gamma m v = 5.0\,mv$.

③ U^{232} $m_0 = 232.03714\,u \longrightarrow Th$ ($m_0 = 228.02873$)

$+ He$ ($m_0 = 4.00260\,u$)

$1u = 1.6605\times10^{-27}$ Kg

$\Rightarrow \Delta m_0 = 0.00581\,u = 9.64\times10^{-30}$ kg

$\Rightarrow \Delta E = \Delta m_0 c^2 = 8.68\times10^{-13}$ J ≈ 5.4 MeV.

④ 1 TeV proton: $E_0 = m_0c^2 = 938$ MeV

$\Rightarrow \gamma = \dfrac{1\times10^{12}\,eV}{0.938\times10^9\,eV} \approx 10^3 \Rightarrow \beta = \sqrt{1-\dfrac{1}{\gamma^2}} \approx 0.9999995\,c$

$\longrightarrow c$