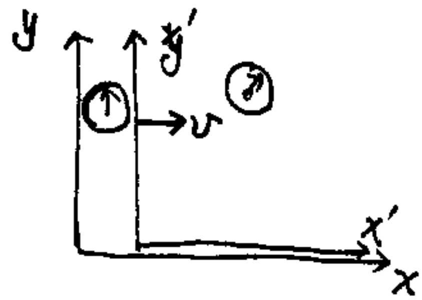
Lect 17 time-dilation and length contraction

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$



§ time-dilation

at $\begin{cases} x = x' = 0 \\ ct = ct' = 0 \end{cases}$ calibration. A clock is kept rest at the origin of F', then the time measured in the rest frame (ie F') is called proper time τ.

$$\Rightarrow \begin{bmatrix} c\tau \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \Rightarrow \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} c\tau \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} t = \gamma\tau \\ \quad > \tau \\ x = \gamma\beta c\tau \end{cases} \qquad \text{or } \tau = \frac{t}{\sqrt{1-\beta^2}}.$$

proper time is the shortest.

§ length contraction:

Imagine a ruler, which at rest in the F', its left and right ends are kept at $x'_L = 0$ and $x'_R = \ell_0$. Now let us measure it length in the F-frame, since it's moving, we have to take the coordinates of the left and right ends simutaneously. Suppose we choose the instance in F such that the left end of the rule passing the origin, i.e $\begin{cases} x_L = 0 \\ t = 0 \end{cases}$.

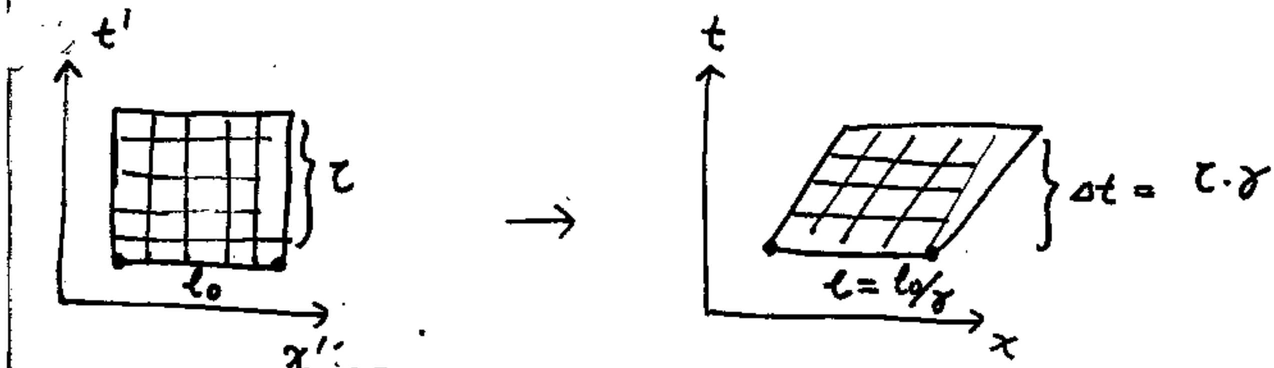
The question is what is the coordinate of x_R at $t=0$.

$$\begin{bmatrix} 0 \\ x_R \end{bmatrix} = \begin{bmatrix} ct \\ x_R \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct' \\ x'_R \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct' \\ l_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ct' \\ l \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ x_R \end{bmatrix} \Rightarrow l_0 = \gamma x_R \text{ or } \boxed{x_R = l = \frac{l_0}{\gamma}}$$

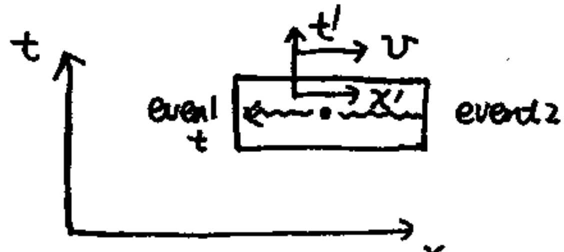
$\boxed{\text{proper length is longest .}}$

The space-time volume $\Delta\tau \Delta l_0 = \Delta t \cdot \Delta l$ which are invariant under Lorentz transframation.



Example: Simultaneily is relative

Suppose in frame F', two events occur the same time, i.e. $\Delta t' = 0$, then at frame F, $c\Delta t = \beta\gamma\Delta x' \Rightarrow \Delta t = \frac{v\Delta x'}{c^2} \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$.



light source at the center of a train, events 1 and 2; the left and right walls receive the light pause.

In frame F', $\Delta t' = 0$, $x'_R - x'_L = \Delta x'$,

In frame F' $\Rightarrow t_R - t_L > 0$.

Example : Casualty is absolute —

two events : $(ds)^2 = (c\Delta t)^2 - (\Delta x)^2$

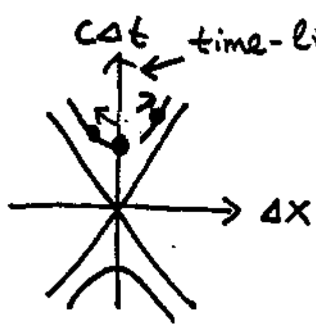
hyperbolic

> 0 time-like

$= 0$ light-like

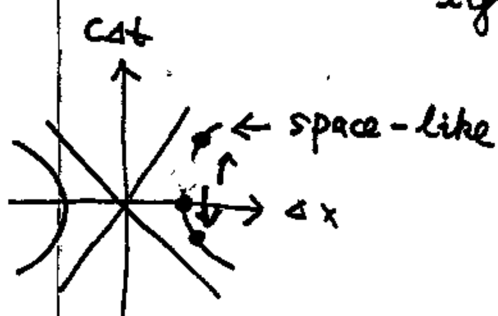
< 0 space-like

if the interval is time-like, (inside light cone



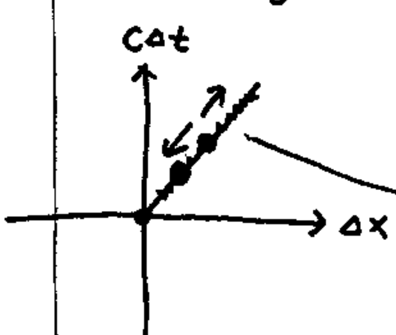
Lorentz transformation shift $(\Delta t, \Delta x)$ along one of the hybolic curves, which does not cross the other branch $\Rightarrow$ Δt does not change sign, i.e. casuality is maintained inside light cone.

branch



$(\Delta t, \Delta x)$ transform along one of the hyperbolic curves, Δt change change sign but ΔX doesn't. — relativity of simultaneity.

if $\Delta X = ct$. light-like



$$\begin{pmatrix} c\Delta t' \\ \Delta X' \end{pmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta X \end{bmatrix} = \gamma(1-\beta) \begin{bmatrix} \Delta X \\ \Delta x \end{bmatrix}$$

$(c\Delta t, \Delta x)$ cannot pass the origin.

Ex: life time of μ. in the rest frame, the life time $\tau = 2.20\,\mu s$

if μ is traveling at $v = 0.6c$, then its life time $\Delta t = \frac{\tau}{\sqrt{1-\beta^2}} = \frac{2.20}{0.8}$

$= 2.8 \times 10^{-6}\,s$, and the distance it travels is $d = v\Delta t = 500\,m$.

Ex: a car travels at 100 km/h ($\approx 30\,m/s$), and the driver spends 10 s, (proper time).

on the ground $\Delta t = \frac{10s}{\sqrt{1-\left(\frac{30}{3\times10^8}\right)^2}} \approx 10s\left(1 + \frac{1}{2}\frac{1}{10^{14}}\right)$

$\approx 10s + 5\times10^{-14}\,s$.

twin paradox:

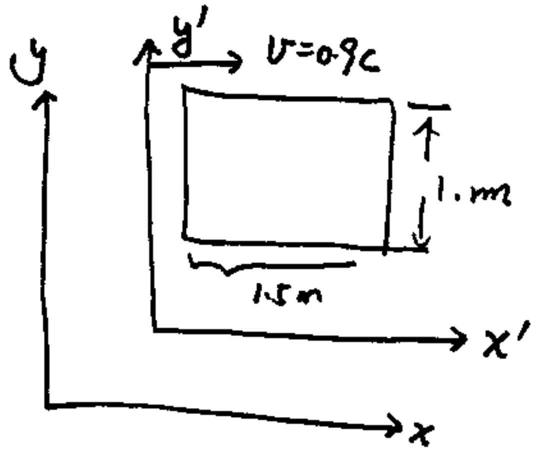
for a star at 100 light year, a space shuttle at $v = 0.999c$. Measured in the earth frame $\Delta t \simeq \frac{100}{0.999} \simeq 100$ year, but in the space shuttle frame $\Delta\tau = \frac{\Delta t}{\gamma} = 100\sqrt{1-(0.999)^2} \approx 4.5\,yr$.

If the astronaut further comes back, then time spent on the earth in total $2\Delta t \simeq 200$ yr. but $2\Delta\tau$ in the shuttle ~ 9 yr.

You might think that relative to the shuttle, the earth also has the speed of $0.999c$, it's not clear whose time is slower.

But the earth frame is special, because shuttle changes motion of direction, thus experiences acceleration. The shuttle frame is not equivalent to the earth frame any more.

Ex in the co-moving frame, a painting $\omega_0 = 1.5m$, $h_0 = 1.0m$,



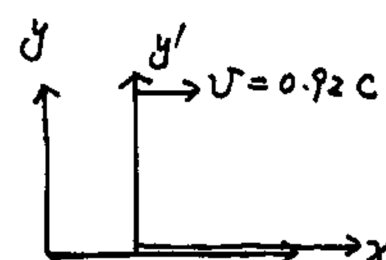
in the lab frame, $\omega = \frac{\omega_0}{\gamma} = \sqrt{1-(\frac{v}{c})^2}\, l_0$

$= \sqrt{0.19} \times 1.5 = 0.65m$

$h = h_0 = 1.0m$

Ex a train with proper length 500m, passing a tunnel with lengh 200m on the ground. Estimate minimum velocity so that the train fits completely the tunnel.

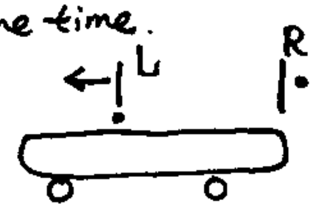
$\gamma = \frac{l_0}{l} = \frac{500}{200} = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta = 0.92.$



But for the train fram, the tunnel is moving $v = -0.92c$, thus its length becomes $\frac{200}{\gamma} = 80m$, how can 80m tunnel hold 500m train?

let us check: in the ground frame, the two events of measuring the left and right ends of the train occurs at the same time.

$$\begin{cases} x_R - x_L = 200m \\ t_R - t_L = 0 \end{cases}$$



However, at the train frame, the tunnel is moving at $v = -0.92c$

$\Rightarrow t'_R - t'_L = -\frac{\beta\gamma}{c}(x_R - x_L) < 0$, when the meaurement of right end take place, the left end measurement has not takenplace yet.

At this instant in the train frame, the tunnel covers $\frac{200\,m}{\gamma}$.
The measument of the left end occurs $t_L' - t_R'$ later with a time delay $\frac{\beta\gamma}{c} \cdot 200m$, during which the left end of the train travels further $\beta^2\gamma \cdot 200m$. Adding together

$$\Rightarrow \underbrace{\left(\frac{1}{\gamma} + \beta^2\gamma\right)}_{200m} = \underbrace{\left[\sqrt{1-\beta^2} + \beta^2\gamma\right]}_{200m} = \left[\gamma(1-\beta^2+\beta^2)\right] 200m = \gamma \cdot 200m = 500m.$$