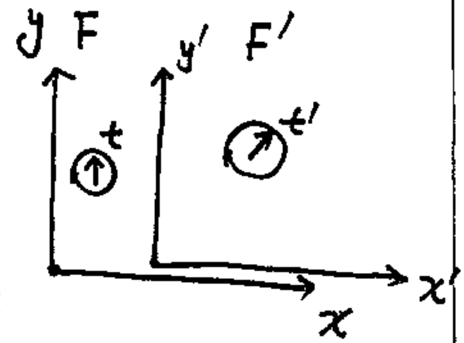
lect 15 Lorentz transformation

two postulates

① The laws of physics have the same form in all inertial frames.

② light velocity c is unique, which does not depend on the source & observers.

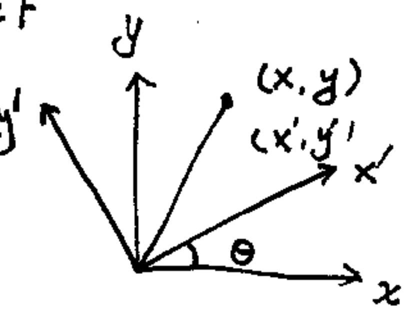
Let us find the transformation of the space-time coordinates of an event (ct, x) in frame F and (ct', x') in another frame F'. We assume that this transformation is linear.



Before doing this, let us try a familiar example of spatial rotation.

The same point's coordinate (x, y) in the frame F and (x', y') in the frame F'.

we know the transformation can be described by 2×2 othogonal matrix



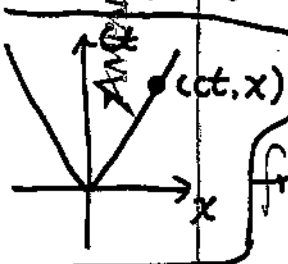
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

the rotation doesn't change $x'^2 + y'^2 = x^2 + y^2$

For Lorentz transformation, we consider the two frames F and F' coincides at $t = t' = 0$. At this case

$$\begin{cases} x' = x = 0 \\ y' = y = 0 \\ z' = z = 0 \\ t' = t = 0 . \end{cases}$$

For another coordinate (ct, x) if $ct = x$, it can be considered as one point along the world line of the light pulse emitted from the origin. Due to the invariance of light velocity, the same event in another frame should be (ct', x'). Thus $(ct)^2 - x^2 = (ct')^2 - x'^2$.



We define the distance square as $s^2 = (ct)^2 - x^2 = [ct, x]\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} ct \\ x \end{bmatrix}$.

We have shown that if $s^2 = 0$, then $s'^2 = 0$.

We know that ds' and ds are the same order & $(ds)^2 = 0$ iff $(ds')^2 = 0$ $\Rightarrow$ $(ds')^2 = a(ds)^2$.

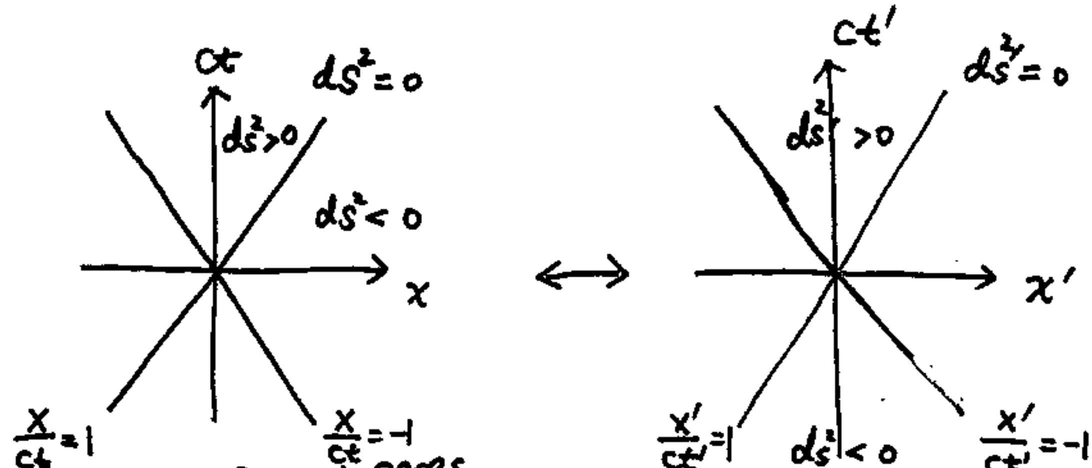
The coefficient "a" only depends on the relative motion between two frames, and only depends on the absolute value of the velocity.

"a" should not depend on "time and coordinate", otherwise it's against the the homogeneity of space-time. "a" should not depend on the direction of the relative motion, otherwise it's against the isotropy of space-time.

We can exchange the positions of F and F'. Relative to F', F is moving with velocity $-V$, thus a does not change. We can also write $(ds)^2 = a(ds')^2$. $\Rightarrow (ds)^2 = a(ds')^2 = a^2(ds)^2$

$\Rightarrow a^2 = 1$ or $a = \pm 1$.

"a" can either 1 or -1, but not 1 for some cases and -1 for other cases, otherwise "a" should be able to take some values between 1 and -1 due to continuity, which is not allowed. Based on physical reasons, we take $a=1$.

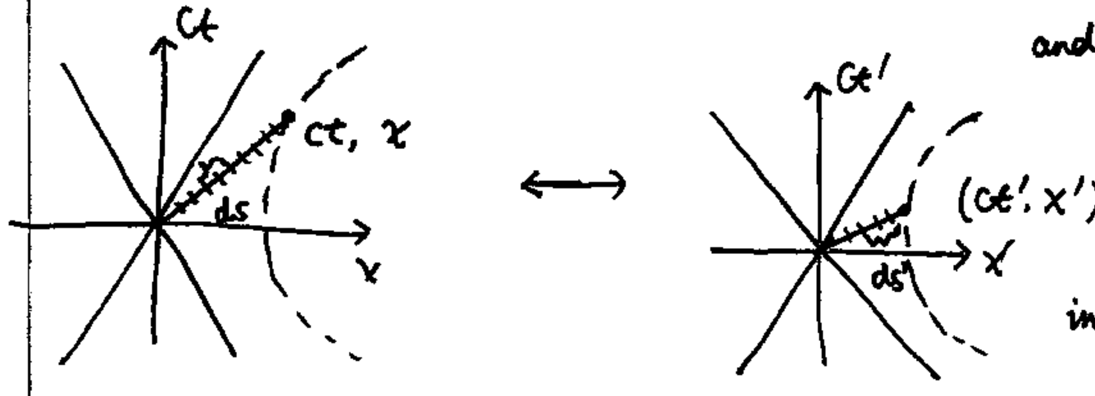


Proof:

if we take $a=-1$, for all cases we will be in trouble we consider three frames F F' and F'', then $(ds)^2 = a(ds')^2 = a \cdot a(ds'')^2$

$\Rightarrow (ds)^2 = (ds'')^2$ which is against the choice of $a=-1$.

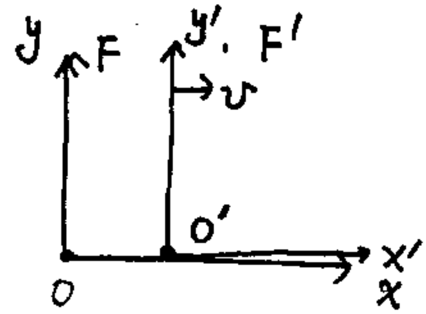
for the finite intervals S and S' $\Rightarrow$ we cut the lines from 0 to (ct, x) and 0 to (ct', x') into many small segaments, apply $ds = ds'$ and integrate $\Rightarrow S^2 = S'^2$



Thus Lorentz tranformation maintains $(ct)^2 - x^2$ invarinat,

thus it can be parametrized as

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} ch\theta & -sh\theta \\ -sh\theta & ch\theta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$



The origin of F', i.e O', its $x' \equiv 0$, and it's motion in F is $\frac{v}{c} ct = x$. $\Rightarrow$

$$0 = x' = -ct\, sh\theta + x\, ch\theta \Rightarrow \tanh\theta = \frac{x}{ct} = \frac{v}{c} = \beta$$

$$\Rightarrow \begin{cases} ch\theta = \dfrac{1}{\sqrt{1-\beta^2}} = \gamma \\ sh\theta = \dfrac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta \end{cases} \Rightarrow \boxed{\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}}$$