Lect 8: Free fermion

Consider free fermion Dirac in 1+1 dimension

$$\Psi(t,x) = \begin{pmatrix} \psi(t,x) \\ \bar{\psi}(t,x) \end{pmatrix} \leftarrow \text{left and right movers}$$

We could impose periodic or anti-periodic boundary condition around the cylinder, i.e. $\psi(t, x+L) = \pm \psi(t,x)$ — $+ \to$ Ramond sector
$- \to$ Neveu-Schwarz sector

$$S[\Psi] = \frac{i}{4\pi} \int_{cyl} \Psi^{+}(t,x)\gamma^0 [\gamma^0 \partial_0 + \gamma^1 \partial_1] \Psi(t,x).$$

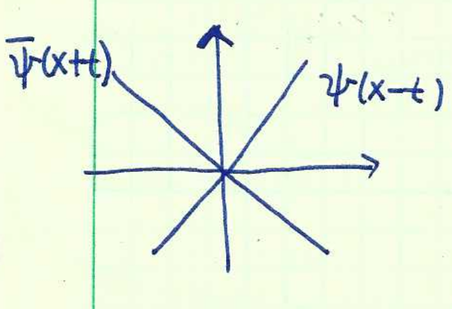
we use $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ $\Rightarrow \alpha = \gamma^0\gamma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\Rightarrow S[\Psi] = \frac{1}{4\pi} \int [\psi(\partial_0 + \partial_1)\psi + \bar{\psi}(\partial_0 - \partial_1)\bar{\psi}]\, dt\, dx$ hence $\psi, \bar{\psi}$ are chiral eigenbasis

$$\longrightarrow \quad \frac{1}{2\pi} \int \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \quad dx\, dt \qquad \partial = \tfrac{1}{2}(\partial_0 + \partial_1), \quad \bar{\partial} = \tfrac{1}{2}(\partial_0 - \partial_1).$$

where we consider $\psi, \bar{\psi}$ as real Majorana field.

EOM $\Rightarrow \bar{\partial}\psi = \partial\bar{\psi} = 0 \Rightarrow \psi = \underbrace{\psi(\;)}_{x-t}$ and $\bar{\psi} = \overbrace{\bar{\psi}(\;)}^{x+t}$.



transform to complex ~~field~~ coordinate

$$\psi(z) = \left(\frac{\partial z}{\partial x}\right)^{1/2} \psi(t,x) = \left(\frac{2\pi i z}{L}\right)^{1/2} \psi(t,x)$$

$$z = e^{\frac{2\pi}{L}(\tau + ix)}$$

Since $x \to x + L$ corresponds to $z \to z e^{2\pi i}$, hence

$$\psi(2\pi i z) = e^{i\pi} \left(\frac{2\pi i z}{L}\right)^{1/2} \psi(t, x+L) = \begin{cases} -\psi(z) & \text{anti-periodic (Ramond)} \\ \psi(z) & \text{periodic (Neveu Schwaz)} \end{cases}$$

Hence, we have the following decomposition

$$\psi(z) = \begin{cases} \sum_{n\in \mathbb{Z}} b_n z^{-n-1/2} & \text{(Ramond)} \\ \sum_{n \in \mathbb{Z}+1/2} b_n z^{-n-1/2} & \text{(N-S)} \end{cases} \to b_n = \oint_C \psi(z) z^{n-1/2} \frac{dz}{2\pi i}$$

Cannoical quantization gives to anti-commutation

$$b_m b_n + b_n b_m = \delta_{m+n,0}$$

and $b_n^2 = \frac{1}{2} \delta_{n,0}$

Fock spaces: we set b_n annihilaters for $n > 0$
creaters for $n < 0$.

In the Ramond sector, there exist b_0, but b_0 is not a zero mode such that we can define $b_0 |\lambda\rangle = \lambda |\lambda\rangle$. It's not an annihilater since $b_0(b_0|\lambda\rangle) = \frac{1}{2}|\lambda\rangle$. Hence b_0 is a creater.

Ramond sector

$$\begin{array}{ccccc} \vdots & \vdots & & & \vdots \\ & & & & b_{-3/2}\, b_{-1/2} |NS\rangle \\ b_{-2}|R\rangle & b_{-2} b_0 |R\rangle & \text{or} & & b_{-3/2} |NS\rangle \\ b_{-1}|R\rangle & b_{-1} b_0 |R\rangle & & & b_{-1/2}|NS\rangle \\ |R\rangle & b_0|R\rangle - \text{degenerate} & & & |NS\rangle \end{array}$$

If we think $|R\rangle$ is the vacuum, it would be a problem

$$\left.\begin{array}{l}\lim_{z\to 0} \psi(z)|R\rangle = \lim_{z\to 0} [\, b_0|R\rangle z^{-1/2} + b_1|R\rangle z^{1/2} + \cdots \,] \\ b_0|R\rangle \neq 0, \text{ but } z^{-1/2} \to \infty \end{array}\right\} \text{No limit.}$$

Then we use

$$\lim_{z\to 0} \psi(z)|NS\rangle = \lim_{z\to 0} \sum_{n\in\mathbb{Z}+1/2} b_n|NS\rangle z^{-n-1/2} = \lim_{z\to 0} b_{-1/2}|NS\rangle + b_{-3/2}|NS\rangle z + \cdots$$

$$= b_{-1/2}|NS\rangle$$

We denote $\boxed{|0\rangle = |NS\rangle \quad \text{and } |\psi\rangle = b_{-1/2}|0\rangle.}$

Radial-ordering: now let's consider the NS secter.

Let the parity of $A(z)$ be denoted $\bar{A} = \begin{cases} 0 & \text{bosonic} \\ 1 & \text{fermionic} \end{cases}$

Define

$$R\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z|>|w| \\ (-)^{\bar{A}\bar{B}} B(w)A(z) & \text{if } |z|<|w| \end{cases}$$

OPE:

$$R\{\psi(z)\psi(w)\} = \sum_{n\in\mathbb{Z}+1/2} \phi_n(w)(z-w)^{-n-1/2}$$

$$\lim_{w\to 0} R\{\psi(z)\psi(w)\}|0\rangle = \sum_{n\in\mathbb{Z}+1/2} \lim_{w\to 0}\phi_n(w)|0\rangle\, z^{-n-1/2}$$

$$\psi(z)\, b_{-1/2}|NS\rangle = \psi(z)|\psi\rangle = \sum_{n\in\mathbb{Z}+1/2} |\phi_n\rangle z^{-n-1/2}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}+1/2} b_n|\psi\rangle z^{-n-1/2} = \sum_{n\in\mathbb{Z}+1/2} |\phi_n\rangle z^{-n-1/2}$$

$$|\phi_n\rangle = b_n|\psi\rangle = b_n b_{-1/2}|0\rangle$$

$n > 1/2 \quad |\phi_n\rangle = b_n b_{-1/2}|0\rangle = -b_{-1/2} b_n|0\rangle = 0 \Rightarrow \phi_n(w) = 0$ for $n > 1/2$.

$n = 1/2 \quad |\phi_{1/2}\rangle = b_{1/2} b_{-1/2}|0\rangle = (-b_{-1/2} b_{1/2} + 1)|0\rangle = |0\rangle$

$$\phi_{1/2}(w) = 1$$

$$\Rightarrow \boxed{R\{\psi(z)\psi(w)\} \sim \frac{1}{z-w} = -R\{\psi(w)\psi(z)\}}$$

Or we directly check:

$$\psi(z) = \sum_{n\in\mathbb{Z}+1/2} b_n z^{-n-1/2}, \quad \psi(w) = \sum_{r\in\mathbb{Z}+1/2} b_r w^{-r-1/2}$$

if $|z| > |w|$,

$$\psi(z)\psi(w) = \sum_{n,r} b_n b_r z^{-n-1/2} w^{-r-1/2}$$

$$= \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}+1/2} b_r b_{n-r} z^{-r-1/2} w^{-n+r-1/2}$$

$$= \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}+1/2} :b_r b_{n-r}: z^{-r-1/2} w^{-n+r-1/2} + \sum_{r\geq 1/2} z^{-r-1/2} w^{r-1/2} \quad \leftarrow \text{set } n=0$$

$$\sum_{r\geq 1/2} z^{-r-1/2} w^{r-1/2} = z^{-1} + z^{-2}w + z^{-3}w^2 + \cdots$$
$$= z^{-1}\left[1 + \frac{w}{z} + \left(\frac{w}{z}\right)^2 + \cdots\right] = \frac{1}{z-w}$$

$$\Rightarrow \psi(z)\psi(w) = :\psi(z)\psi(w): + \frac{1}{z-w}$$

we define ordering

$$:b_m b_n: = \begin{cases} b_m b_n & \text{if } m \leq -1/2 \\ -b_n b_m & \text{if } m \geq 1/2 \end{cases}$$

Check $|z| < |w|$, we have

$$R\{\psi(z)\psi(w)\} \sim \frac{1}{z-w}.$$

(*) Stress-energy tensor:

We take $T(z) = -\alpha :\psi(z)\partial\psi(z):$, where α is a coefficient to be determined we calculate

$$R\{T(z)\psi(w)\} = -\alpha\, R\{:\psi(z)\partial\psi(z):\psi(w)\}$$

$$= -\alpha :\psi(z)\partial\psi(z):\psi(w) - \alpha :\psi(z)\partial\psi(z):\psi(w)$$

extra "-"

$$= \alpha \frac{1}{z-w}\partial\psi(z) - \alpha\, \partial\frac{1}{z-w}\,\psi(z)$$

$$= \frac{\alpha}{(z-w)^2}\psi(z) + \frac{\alpha}{(z-w)}\partial\psi(z)$$

Compare $T(z)\psi_h(w) = \frac{h\psi_h(w)}{(z-w)^2} + \frac{\partial\psi_h(w)}{z-w}$

$$\psi(z) = \psi(w) + \partial\psi(w)(z-w)$$

$$\Rightarrow R\{T(z)\psi(w)\} = \frac{\alpha}{(z-w)^2}\psi(w) + \frac{2\alpha}{z-w}\partial\psi(w)$$

hence we set $\alpha = 1/2$, since the scaling dimension of ψ is $1/2$

Then $\boxed{T(z) = -1/2 :\psi(z)\partial\psi(z):}$

$$\boxed{R\{T(z)\psi(w)\} = \frac{1/2}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial\psi(w)}$$

Then $R\{T(z)T(w)\} = 1/4\, R\{:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):\}$

$$:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): = \partial_w\frac{1}{z-w}\partial_z\frac{1}{z-w} = \frac{-1}{(z-w)^4}$$

$$:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): = -\frac{1}{z-w}\partial_z\partial_w\frac{1}{z-w} = \frac{2}{(z-w)^4}$$

$$:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): = \partial_w\frac{1}{z-w} :\partial\psi(z)\psi(w):$$

$$= \frac{1}{(z-w)^2} :\partial\psi(w)\psi(w): + \frac{:\partial^2\psi(w)\psi(w):}{(z-w)}$$

$$:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): \; = -\frac{1}{z-w} :\partial\psi(z)\partial\psi(w):$$

$$= -\frac{1}{z-w} :\partial\psi(w)\partial\psi(w): \; - \; :\partial\psi(w)\partial\psi(w):$$

$$:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): \; = \partial_z \frac{1}{z-w} :\psi(z)\partial\psi(w):$$

$$= \frac{-1}{(z-w)^2} :\psi(w)\partial\psi(w): \; - \frac{1}{(z-w)} :\partial\psi(w)\partial\psi(w):$$

$$:\psi(z)\partial_z\psi(z)::\psi(w)\partial_w\psi(w): \; = -\partial_z\partial_w \frac{1}{z-w} \psi(z)\psi(w)$$

$$= \frac{2}{(z-w)^3}\left[(z-w)\partial\psi(w) + \frac{1}{2}(z-w)^2\partial^2\psi + \frac{1}{6}(z-w)^3\partial^3\psi\right]\psi(w)$$

$$= \frac{2}{(z-w)^2} :\partial\psi(w)\psi(w): + \frac{1}{z-w} :\partial^2\psi\,\psi: \qquad -\psi\partial^2\psi \downarrow$$

Add together: $\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}[4:\partial\psi\,\psi:] + \frac{2}{z-w}[:\partial^2\psi\,\psi - \partial\psi\partial\psi]$

$$\Rightarrow R[T(z)T(w)] = \frac{1/4}{(z-w)^4} + \frac{-1}{(z-w)^2}(:\psi\partial\psi:] - \frac{1/2}{z-w}\partial[\psi\partial\psi]$$

$$= \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$\downarrow$

$1/4 = \frac{c}{2} \Rightarrow c = 1/2.$ Hence free fermion is a CFT with $c=1/2$

We express $T = \sum_n L_n z^{-n-2}$

$$\psi(z) = \sum_{n\in \mathbb{Z}+1/2} b_n z^{-n-1/2} \quad \text{in the NS sector}$$

$$\partial\psi(z) = -\sum_{n\in \mathbb{Z}+1/2} (n+1/2)\, b_n z^{-n-3/2}$$

$$T = -1/2 :\psi\partial\psi: \;=\; 1/2 \sum_{n\in\mathbb{Z}} z^{-n-2} \sum_{r\in\mathbb{Z}+1/2} (n-r+1/2) :b_r b_{n-r}:$$

Hence $\boxed{L_n = 1/2\left(\sum_{r\le -1/2} (n-r+1/2) :b_r b_{n-r}: - \sum_{r\ge 1/2} (n-r+1/2) :b_{n-r} b_r:\right)}$

- Now we calculate the energy

$$|0\rangle = |NS\rangle$$

$$L_0|NS\rangle = -1/2\left[\sum_{r\le -1/2} (r-1/2) :b_r b_{-r}:\right] + 1/2 \sum_{r\ge 1/2} (r-1/2) :b_{-r} b_r:] \; |NS\rangle = 0.$$

So far, we define normal ordering in the NS sector. We cannot directly use it for the Ramond sector.

Let us prove a general relation for bothe Ramond and NS

$$\boxed{\sum_{r=0}^{\infty} (r+1) [b_{m-r} b_{n+r} + b_{n-r-2} b_{m+r+2}] = \frac{(2m+1)(2m+3)}{8} \delta_{m+n,0} + 2 L_{m+n}}$$

where m, n take integer values for the Ramond sector

half-integer values for the N-S sector