Lect 7: - Nested Bethe Ansatz

Spin-1/2 fermion model has two conserved quantities: the total particle number N and the total spin S_z. Correspondinly, we have two sets BA quantum numbers and two sets of BA equations

$$\boxed{\begin{array}{lll} ① \; e^{ik_jL} = \prod_{\alpha=1}^{M} \dfrac{k_j - \Lambda_\alpha + ic/2}{k_j - \Lambda_\alpha - ic/2} & \text{for each } k_j \;\; j=1,2,\cdots N & \text{(particle number)} \\[2ex] ② \; \prod_{j=1}^{N} \dfrac{k_j - \Lambda_\alpha - ic/2}{k_j - \Lambda_\alpha + ic/2} = -\prod_{\beta=1}^{M} \dfrac{\Lambda_\beta - \Lambda_\alpha - ic}{\Lambda_\beta - \Lambda_\alpha + ic} & \alpha = 1, \cdots M & \text{(\# of spin up)} \end{array}}$$

k_j $(j=1,2\cdots N)$ and Λ_α $(\alpha=1,\cdots M)$ are two sets of physical quantities that we are going to solve.

Eq ① $\Rightarrow$

$$k_jL = 2\pi \cdot \text{Integer} + \sum_{\alpha=1}^{M} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)}$$

$$= 2\pi \cdot \text{integer} + \sum_{\alpha=1}^{M} \pi - 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c}$$

$$\boxed{\begin{array}{l} k_jL = 2\pi I_j - 2\sum_{\alpha=1}^{M} \tan^{-1} \dfrac{2(k_j-\Lambda_\alpha)}{c} \qquad (j=1,2,\cdots N) \\[2ex] I_j: \text{ integer if } M \text{ is even;} \\ \quad\;\; \text{half integer if } M \text{ is odd.} \end{array}}$$

Eq ② $\Rightarrow$ $$\sum_{j=1}^{N} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)} = 2\pi \cdot \text{integer} + \pi + \sum_{\beta=1}^{M} 2\tan^{-1} \frac{c}{\Lambda_\beta - \Lambda_\alpha}$$

$$\sum_{j=1}^{N} \pi - \sum_{j=1}^{N} 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi\,[\otimes]\,[\text{half integer}] + \sum_{\beta=1}^{M} \pi - \sum_{\beta=1}^{M} 2\tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$-\sum_{j=1}^{N} 2\tan^{-1}\frac{2(k_j-\Lambda_\alpha)}{c} = 2\pi\left(\text{half integer} + \frac{N-M}{2}\right) - \sum_{\beta=1}^{M} 2\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c}$$

$$\boxed{2\sum_{j=1}^{N} \tan^{-1}\frac{2(k_j-\Lambda_\alpha)}{c} = -2\pi J_\alpha + 2\sum_{\beta=1}^{M}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} \qquad \alpha=1,\cdots M}$$

where $J_\alpha = \begin{cases} \text{integer}, & \text{if } N-M \text{ is odd} \\ \text{half-integer} & \text{if } N-M \text{ is even} \end{cases}$

Ground state Solution:

① First consider the limit $c\to\infty$, $N=$ even, $m=$ odd

a) $k_j L = 2\pi I_j + 2\sum_{\beta=1}^{M}\tan^{-1} 2\Lambda_\beta/c \qquad j=1,\cdots N$

b) $-2N\tan^{-1}\frac{2\Lambda_\alpha}{c} = 2\pi J_\alpha + 2\sum_{\beta=1}^{M}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} \qquad \alpha=1,\cdots M$

I_j: half integer. J_α: integer

b) $\Rightarrow \quad -2N\sum_{\alpha=1}^{M}\tan^{-1}\frac{2\Lambda_\alpha}{c} = -2\pi\sum_{\alpha=1}^{M} J_\alpha \quad \leftarrow \quad \sum_{\alpha,\beta}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} = 0$

hence $\quad 2\sum_{\alpha=1}^{M}\tan^{-1}\frac{2\Lambda_\alpha}{c} = \frac{2\pi}{N}\sum_{\alpha=1}^{M} J_\alpha$

$$k_j = \frac{2\pi}{L}(I_j + a), \qquad \text{with } a = -1/N\sum_{\alpha=1}^{M} J_\alpha$$

$$E = \sum_{j=1}^{N} k_j^2 = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{N}(I_j+a)^2$$

Ground state: when $a=0$, and I_j a is closely packed around 0 symmetrically.

* $a=0$ can beviewed as the center of mass memontum.

$(I_j+a)^2 \propto$ the ith particle kinetic energy.

$a=0 \Rightarrow J_\alpha$ takes ~~vaule~~ symmetrically with respect to zero

from b) $\Rightarrow$ Λ_α and $-\Lambda_\alpha$ appear in pairs.

2) $c \neq \infty$, in the ground state I_j still takes the value as the case of $c=\infty$. J_α distributes symmetrically around the origin, and Λ and $-\Lambda$ paired.

$$2\pi I_j = L k_j + \sum_\alpha 2\tan^{-1}\frac{2k_j - 2\Lambda_\alpha}{c} = L k_j + \sum_\alpha \tan^{-1}\frac{2k_j - 2\Lambda_\alpha}{c} + \tan^{-1}\frac{2k_j + 2\Lambda_\alpha}{c}$$

$$2\pi I_j = L k_j + \sum_\alpha \tan^{-1}\frac{4k_j c}{c^2 + 4\Lambda_\alpha^2}$$

where Λ_α^2 increases, k_j increases $\Rightarrow$ where Λ_α closely packed around zero, we arrive at the ground state

we consider the ground state in the case $N, L, M \to \infty$, but N/L, M/L fixed.

$I_{j+1} - I_j = 1$, $\quad J_{\alpha+1} - J_\alpha = 1$

define
$$\boxed{\begin{aligned} L\rho(k)\,dk &= \#\text{ of } k_j \text{ in the interval } k \to k+dk \\ L\sigma(\Lambda)\,d\Lambda &= \#\text{ of } \Lambda_\alpha \text{ in the interval } \Lambda \to \Lambda + d\Lambda. \end{aligned}}$$

$f = I/L$ and $g = J/L$

$\Rightarrow df/dk = \rho(k)$ and $dg/d\Lambda = \sigma(\Lambda)$

From $k_j L = 2\pi I_j + \sum_{\alpha=1}^{M} \theta(2k_j - 2\Lambda)$ where $\theta(x) = -2\tan^{-1} x/c$

$$\to \boxed{k = 2\pi f + \int_{-B}^{B} \theta(2k - 2\Lambda)\,\sigma(\Lambda)\,d\Lambda}$$

$$\frac{d}{dk} \rightarrow \boxed{2\pi \rho(k) = 1 + \int_{-B}^{B} \frac{4c\,\sigma(\Lambda)}{c^2 + 4(k-\Lambda)^2}\, d\Lambda}$$

by using $d\theta/dx = -2\, c/(c^2+x^2)$

$$2 \sum_{j=1}^{N} \tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = -2\pi J_\alpha + 2 \sum_{\beta=1}^{M} \tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$\sum_{j=1}^{N} \theta(2\Lambda_\alpha - 2k_j) = -2\pi J_\alpha + \sum_{\beta=1}^{M} \theta(\Lambda_\alpha - \Lambda_\beta)$$

$$\int_{-Q}^{Q} \theta(2\Lambda - 2k)\, \rho(k)\, dk = -2\pi g + \int_{-B}^{B} \theta(\Lambda - \Lambda')\, \sigma(\Lambda')\, d\Lambda'$$

$$\frac{d}{d\Lambda} \Rightarrow \boxed{2\pi \sigma(\Lambda) = -\int_{-B}^{B} \frac{2c\,\sigma(\Lambda')}{c^2 + (\Lambda-\Lambda')^2}\, d\Lambda' + \int_{-Q}^{Q} \frac{4c\,\rho(k)\, dk}{c^2 + 4(k-\Lambda)^2}}$$

Also the constraints

$$\boxed{\frac{N}{L} = \int_{-Q}^{Q} \rho(k)\, dk, \quad \frac{M}{L} = \int_{-B}^{B} \sigma(\Lambda)\, d\Lambda}$$

we can solve ρ, σ, Q, B, then

$$\frac{E}{L} = \int_{-Q}^{Q} k^2 \rho(k)\, dk$$