OPE

Lect 6: Free bosons (II):

Define the true vacuum $|p=0\rangle$ since it's energy $E_p = \frac{1}{2}p^2 = 0$.

Given a field $\psi(z)$, there exist a quantum state $|\psi\rangle$ defined by $\boxed{|\psi\rangle = \lim_{z\to 0} \psi(z)|0\rangle.}$ Since $z = e^{\frac{2\pi}{L}(\tau + ix)}$, $z \to 0$ corresponds to $\tau \to -\infty$.

(state-field correspondence)

$a_0|0\rangle = a_1|0\rangle = \cdots$

Example: $\partial\varphi(z) = \sum_n a_n z^{-n-1}$

$$\Rightarrow \lim_{z\to 0} \partial\varphi(z)|0\rangle = \lim_{z\to 0}\sum_n a_n z^{-n-1}|0\rangle = \lim_{z\to 0}\sum_{n\leq -1} a_n z^{-n-1}|0\rangle$$

$$= a_{-1}|0\rangle, \quad \rightsquigarrow \text{denoted } |\partial\varphi\rangle = a_{-1}|0\rangle.$$

- $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2} :\partial\varphi(z)\,\partial\varphi(z):$

$$\lim_{z\to 0} T(z)|0\rangle = \lim_{z\to 0}(L_{-2} + L_{-1}z^{-1} + L_0 z^{-2} + \cdots)|0\rangle$$

$$L_{-1}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{-r-1}: = \frac{1}{2}(\cdots a_{-2}a_1 + a_{-1}a_0 + a_1 a_0 + a_{-2}a_1 + \cdots)|0\rangle = 0$$

$$L_n|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{-r+n}: = \frac{1}{2}(\cdots a_{-2}a_{2+n} + a_{-1}a_{1+n} + a_0 a_n + a_{n+1}a_1 + a_{n-2}a_2 + \cdots)|0\rangle = 0$$

$(n \geq 0)$

$$L_{-2}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{r-2}:|0\rangle = \frac{1}{2}(\cdots a_{-2}a_0 + a_{-1}a_{-1} + a_2 a_0 + a_3 a_1 + \cdots)|0\rangle$$

$$= \frac{1}{2} a_{-1}^2 |0\rangle$$

Hence $\boxed{T(z)|0\rangle \longrightarrow \frac{1}{2}a_{-1}^2|0\rangle.}$

- $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $\Rightarrow \partial^2\varphi(z) = \sum_{n\in\mathbb{Z}} -(n+1)\, a_n\, z^{-n-2}$

$$\partial^2\varphi(z)\,|0\rangle = a_{-2} + \underbrace{0\, a_{-1} z^{-1} + (-1)\, a_0 z^{-2} + (-2)\, a_1 z^{-3} + \cdots}_{=0} \;|0\rangle$$

$$= a_{-2}|0\rangle$$

- Operator product expansion. (OPE)

Check $[\partial\varphi(z), \partial\varphi(w)] = \sum_{m\in\mathbb{Z}} [a_m, \partial\varphi(w)]\, z^{-m-1}$

$$[a_m, \partial\varphi(w)] = \sum_n [a_m, a_n]\, w^{-n-1} = m\, w^{m-1}$$

$$\Rightarrow [\partial\varphi(z), \partial\varphi(w)] = \sum_{m\in\mathbb{Z}} m\, w^{m-1} z^{-m-1} = \frac{1}{zw} \underbrace{\sum_{m\in\mathbb{Z}} m \left(\frac{w}{z}\right)^m}_{\text{ill-defined}}$$

To remove this, we define time-ordering.

According to $z = e^{2\pi(\tau + ix)}$, hence, this corresponds to ordering along the radial direction. — Radial ordering.

$$\boxed{R\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z|>|w|, \ z \text{ is later} \\ B(w)A(z) & \text{if } |z|<|w|, \ w \text{ is later.} \end{cases}}$$

We interpret $A(z)B(w)|c\rangle$ as $|c\rangle$ is a state at $\tau = -\infty$, then B acts at tim $\tau = w$, and then A acts at time $\tau = z$.

Then

$$R\{[\partial\varphi(z), \partial\varphi(w)]\} = R\{\partial\varphi(z)\,\partial\varphi(w)\} - R\{\partial\varphi(w)\,\partial\varphi(z)\}$$

$$= 0$$

Let us look at the radial ordering, more carefully: At $|z|>|\omega|$,

$$R\{\partial\varphi(z)\partial\varphi(\omega)\} = \partial\varphi(z)\partial\varphi(\omega) = \sum_{r,s\in\mathbb{Z}} a_r a_s\, z^{-r-1}\omega^{-s-1}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} a_r a_{n-r}\, z^{-r-1}\omega^{-n-1+r}$$

if $n\neq 0$, since $[a_r, a_{n-r}]=0$, $\quad a_r a_{n-r} = a_{n-r}a_r = :a_r a_{n-r}:$

if $n=0$: $\quad a_r a_{-r} = :a_r a_{-r}:$ for $r\leq -1$

$$a_r a_{-r} = a_{-r}a_r + [a_r, a_{-r}] = :a_r a_{-r}: + r \quad :f\ r\geq 0$$

$$R\{\partial\varphi(z)\partial\varphi(\omega)\} = \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}:\, z^{-r-1}\omega^{-n-1+r} + \sum_{r=0}^{\infty} r\, z^{-r-1}\omega^{-1+r}$$

$$= :\partial\varphi(z)\partial\varphi(\omega): + \frac{1}{z^2}\sum_{r=0}^{\infty} r\left(\frac{\omega}{z}\right)^{r-1} \leftarrow \frac{1}{z^2}\frac{1}{(1-\omega/z)^2}$$

$$\frac{1}{1-x} = 1+x+x^2+\cdots \Rightarrow \left(\frac{1}{1-x}\right)^2 = (1+x+x^2+\cdots)^2$$
$$= 1+2x+3x^2+\cdots.$$

$$\Rightarrow R\{\partial\varphi(z)\partial\varphi(\omega)\} = :\partial\varphi(z)\partial\varphi(\omega): + \frac{1}{(z-\omega)^2} \quad (\text{at } |z|>|\omega|)$$

At $|z|<|\omega|$, then we have

$$R\{\partial\varphi(z)\partial\varphi(\omega)\} = \partial\varphi(\omega)\partial\varphi(z) = :\partial\varphi(\omega)\partial\varphi(z): + \frac{1}{(\omega-z)^2}$$

It's easy to prove $:\partial\varphi(z)\partial\varphi(\omega): = :\partial\varphi(\omega)\partial\varphi(z):$

$$\Rightarrow \boxed{R\{\partial\varphi(z)\partial\varphi(\omega)\} = \frac{1}{(z-\omega)^2} + :\partial\varphi(z)\partial\varphi(\omega):}$$

$\rightarrow$ Taylor expansion $z\to\omega$

$$\boxed{R\{\partial\varphi(z)\partial\varphi(\omega)\} = \underbrace{\frac{1}{(z-\omega)^2} + :\partial\varphi(\omega)\partial\varphi(\omega): + :\partial^2\varphi(\omega)\partial\varphi(\omega):(z-\omega)+\cdots}_{\text{Laurent series expansion}}}$$

The normal ordering can be expressed as

$$:\partial\varphi(w)\,\partial\varphi(w): \;=\; \oint_w \frac{R\{\partial\varphi(\tau)\,\partial\varphi(w)\}}{\tau - w}\,\frac{d\tau}{2\pi i}\,.$$

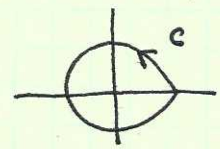
Generally: $R\{A(z)B(w)\} =$ singular terms at $z \to w$ $+ \;:A(z)B(w):$

☆ By using OPE, we can calculate the commutator between the Fourier component of $A(z)$ and $B(w)$. Suppose we have

$$\boxed{R\{A(z_1)B(z_2)\} = \frac{C(z_2)}{(z_1-z_2)^2} + \frac{D(z_2)}{(z_1-z_2)} + O(1)}$$

$$A_m = \oint A(z)\, z^m \frac{dz}{2\pi i} \quad\longrightarrow\quad A(z) = \sum_{n\in\mathbb{Z}} A_n\, z^{-n-1}$$

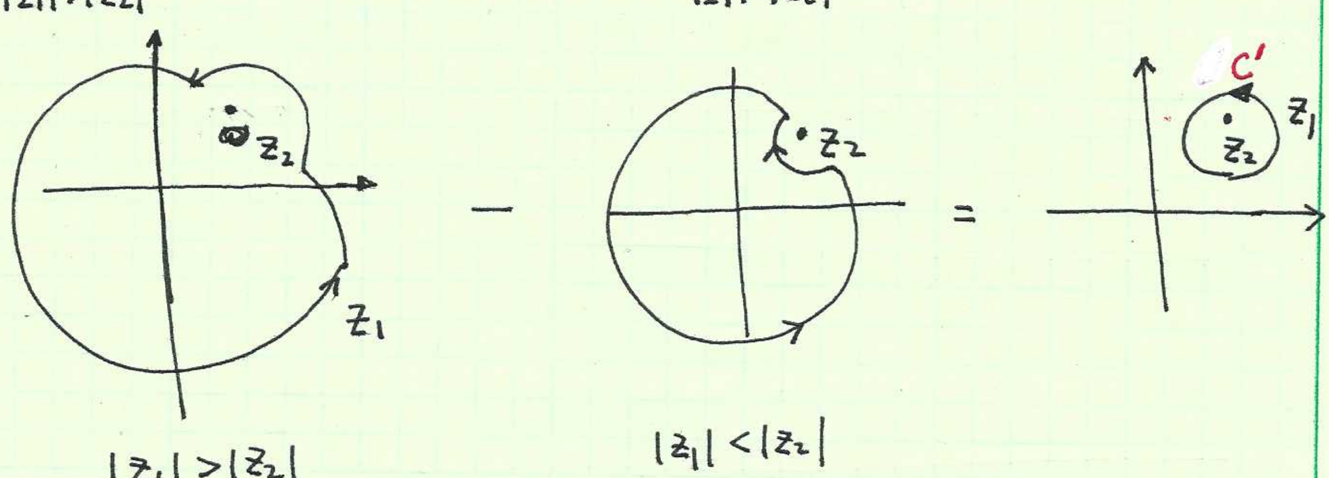
$$B_n = \oint B(z)\, z^n \frac{dz}{2\pi i} \qquad\qquad B(z) = \sum_{n\in\mathbb{Z}} B_n\, z^{-n-1}$$



$$\Rightarrow [A_m, B_n] = A_m B_n - B_n A_m$$

$$= \oint_c\oint_c \left[A(z_1)\,B(z_2)\, z_1^m z_2^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}\right] - \oint_c\oint_c [B(z_2)A(z_1)]\, z_1^m z_2^n \frac{dz\,dw}{(2\pi i)^2}$$

$$= \underset{|z_1|>|z_2|}{\oint\oint_c} R[A(z_1)B(z_2)]\, z_1^m z_2^n \frac{dz\,dw}{(2\pi i)^2} - \underset{|z_1|<|z_2|}{\oint_c\oint_c} R[A(z_1)B(z_2)]\, z_1^m z_2^n \frac{dz\,dw}{(2\pi i)^2}$$



$$[A_m, B_n] = \oint_{C \text{ of } z_2} \left[\oint_{|z_1|>|z_2|} - \oint_{|z_1|<|z_2|} \right] R\{A(z_1) B(z_2)\} z_1^m z_2^n \frac{dz_1 dz_2}{(2\pi i)^2}$$

$$= \oint_{C \text{ of } z_2} \frac{dz_2}{(2\pi i)} \oint_{C'} \frac{dz_1}{(2\pi i)} \left[\frac{C(z_2)}{(z_1-z_2)^2} + \frac{D(z_2)}{(z_1-z_2)} \right] z_1^m z_2^n$$

$$\oint_{C'} \frac{dz_1}{2\pi i} \left[\frac{C(z_2)}{(z_1-z_2)^2} + \frac{D(z_2)}{(z_1-z_2)} \right] z_1^m = \oint_{C_1} \frac{dz_1}{2\pi i} \left[\frac{m z_2^{m-1} C(z_2)}{(z_1-z_2)} + \frac{D(z_2)\, z_2^m}{(z_1-z_2)} \right]$$

$$= m z_2^{m-1} C(z_2) + D(z_2) z_2^m$$

$$\Rightarrow [A_m, B_n] = \oint_{C \text{ of } z_2} \frac{dz_2}{2\pi i} \left[m z_2^{n+m-1} C(z_2) + D(z_2) z_2^{n+m} \right]$$

$$\boxed{[A_m, B_n] = m C_{n+m} + D_{n+m-1}}$$

If we set $A(z_1) = \partial\varphi(z_1)$, $B(z_2) = \partial\varphi(z_2)$

$$\Rightarrow [a_m, a_n] = m\, \delta_{m+n,0} \quad \leftarrow \quad R(\partial\varphi(z_1)\, \partial\varphi(z_2)] = \frac{1}{(z_1-z_2)^2} + \cdots$$

※ Conversely, if we have the commutation rule, we can also determine the singular part of OPE. For example: Let's write down

$$R\{\partial\varphi(z)\, \partial\varphi(w)\} = \sum_n \psi_n(w) (z-w)^{-n-1},$$ where $\psi_n(w)$

to be determined.

LHS: $\lim_{\omega\to 0} R\{\partial\varphi(z)\,\partial\varphi(\omega)\}|0\rangle = \partial\varphi(z)\lim_{\omega\to 0}\partial\varphi(\omega)|0\rangle$

since $|\partial\varphi\rangle = a_{-1}|0\rangle \Rightarrow \lim_{\omega\to 0} R\{\partial\varphi(z)\partial\varphi(\omega)\}|0\rangle = \partial\varphi(z)\, a_{-1}|0\rangle$

$$= \sum_{n\in\mathbb{Z}} \frac{a_n a_{-1}|0\rangle}{z^{n+1}}$$

RHS: $\lim_{\omega\to 0}\sum_n \dfrac{\psi_n(\omega)}{(z-\omega)^{n+1}}|0\rangle = \sum_n |\psi_n\rangle z^{-(n+1)}$

Compare LHS and RHS, we have $|\psi_n\rangle = a_n a_{-1}|0\rangle$ for all $n\in\mathbb{Z}$

For $n\geq 2$, $a_n a_{-1}|0\rangle = a_{-1}a_n|0\rangle = 0. \Rightarrow \psi_n(\omega) = 0$

$n=1$ $\quad a_1 a_{-1}|0\rangle = (a_{-1}a_1 + 1)|0\rangle = |0\rangle \Rightarrow |\psi_1\rangle = |0\rangle \Rightarrow \psi_1(\omega) =$

$n=0$, $\quad a_0 a_{-1}|0\rangle = a_{-1}a_0|0\rangle = 0 \Rightarrow |\psi_0\rangle = 0 \Rightarrow \psi_0(\omega) = 0$

$n=-1$, $\quad a_{-1}^2|0\rangle = |\psi_{-1}\rangle \Rightarrow \psi_{-1}(\omega) = :\partial\varphi(\omega)\partial\varphi(\omega):$

$$\Rightarrow R\{\partial\varphi(z)\partial\varphi(\omega)\} = \frac{\psi_1(\omega)}{(z-\omega)^2} + \psi_{-1}(\omega) + \psi_{-2}(\omega)(z-\omega) + \cdots$$

$$= \frac{1}{(z-\omega)^2} + :\partial\varphi(\omega)\partial\varphi(\omega):$$

Wick's theorem:

Now we need to calculate more complicated OPE's. We introduce

contraction $$\partial\varphi(z)\,\partial\varphi(w) = \frac{1}{(z-w)^2}$$ ← singular terms.

Then the singular part of OPE can be obtained by taking all possible contractions and normal orderings. For example,

$$R\{T(z)\,\partial\varphi(w)\} = \tfrac{1}{2}\, R\{:\partial\varphi(z)\,\partial\varphi(z):\,\partial\varphi(w)\}$$

$$\sim \tfrac{1}{2}\left[:\partial\varphi(z)\partial\varphi(z):\,\partial\varphi(w) + :\partial\varphi(z)\,\partial\varphi(z):\,\partial\varphi(w)\right]$$

$$= \tfrac{1}{2}\,\frac{:\partial\varphi(z):}{(z-w)^2}\times 2 = \frac{\partial\varphi(z)}{(z-w)^2} \simeq \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{(z-w)} + \text{regular}$$

$$\boxed{R\{T(z)\,\partial\varphi(w)\} \sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}}$$

$$R\{T(z)\,T(w)\} = \tfrac{1}{4}\, R[:\partial\varphi(z)\,\partial\varphi(z)::\,\partial\varphi(w)\,\partial\varphi(w):]$$

$$= \tfrac{1}{4}\,[:\partial\varphi(z)\,\partial\varphi(z)::\,\partial\varphi(w)\partial\varphi(w): + :\partial\varphi(z)\partial\varphi(z)::\,\partial\varphi(w)\partial\varphi(w):$$

$$+ :\partial\varphi(z)\,\partial\varphi(z)::\partial\varphi(w)\,\partial\varphi(w): \quad + :\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\,\partial\varphi(w):$$

$$+ :\partial\varphi(z)\,\partial\varphi(z)::\partial\varphi(w)\,\partial\varphi(w): + :\partial\varphi(z)\partial\varphi(z)::\,\partial\varphi(w)\partial\varphi(w):\,]$$

$$= \frac{1/2}{(z-w)^4} + \frac{:\partial\varphi(z)\,\partial\varphi(w):}{(z-w)^2}$$ ← $\partial\varphi(w)\,\partial\varphi(w) + (z-w)\,\partial^2\varphi\,\partial\varphi(\ldots$

$$\boxed{= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} = R\{T(z)T(w)\}}$$

we used $\partial T(w) = \frac{1}{2}\partial :\partial\varphi(w)\partial\varphi(w): = \frac{1}{2}[:\partial^2\varphi(w)\partial\varphi(w): + :\partial\varphi(w)\partial^2\varphi(w):]$

$= :\partial^2\varphi(w)\partial\varphi(w):$ ← under normal ordering, the sequence is not important

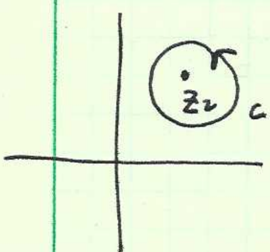
Now let's use the above OPE $T(z)T(w)$ to calculate

$[L_m, L_n]$.

Proof: $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $\Rightarrow L_m = \oint T(z)\, z^{m+1}\frac{dz}{2\pi i}$

$$\Rightarrow [L_m, L_n] = \oint \frac{dz_1}{2\pi i} z_1^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1}\left(T(z_1)T(z_2) - T(z_2)T(z_1)\right)$$

$$= \oint \frac{dz_2}{(2\pi i)} z_2^{n+1} \oint_C \frac{dz_1}{(2\pi i)} z_1^{m+1} R(T(z_1)\,T(z_2))$$



$$\oint_C \frac{dz_1}{2\pi i} z_1^{m+1}\left[\frac{\frac{1}{2}}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{(z_1-z_2)}\right]$$

$$= \frac{1}{2}\frac{(m+1)m(m-1)}{3!}\oint\frac{dz_1}{(2\pi i)}\frac{(z_1-z_2)^3 \cdot z_2^{m-2}}{(z_1-z_2)^4}$$

$$+\ 2(m+1)\oint\frac{dz}{2\pi i}\frac{(z_1-z_2)}{(z_1-z_2)^2} z_2^m\, T(z_2)$$

$$+\ \partial T(z_2)\cdot z_2^{m+1}\oint\frac{dz}{2\pi i}\frac{1}{z_1-z_2}$$

$$= \frac{1}{2} z_2^{m-2}\frac{m(m^2-1)}{3!} + 2z_2^m(m+1) + z_2^{m+1}\partial T(z_2)$$

$$\Rightarrow [L_m, L_n] = \frac{m^3-m}{12}\oint\frac{dz_2}{(2\pi i)} z_2^{m+n-1} + \underbrace{2}_{m+1}\oint\frac{dz_2}{2\pi i} z_2^{m+n+1}\, T(z_2) + \oint\frac{dz_2}{2\pi i}\,\partial T(z_2)\, z_2^{m+n\ldots}$$

$$= \frac{m^3-m}{12}\delta_{m+n,0} + 2(m+1)L_{m+n} + \oint\frac{dz_2}{2\pi i}\sum_{n'}(-n'-2)L_{n'} z_2^{-n'-3}\cdot z_2^{m+n+2}$$

$$\oint \frac{dz_2}{2\pi i} \sum_{n'} (-n'-2)\, L_{n'} \quad z_2^{\,m+n-n'-1}$$

it's non-zero $n' = m+n$, $\Rightarrow (-m-n-2) L_{m+n}$

Combine all terms

$$\Rightarrow \boxed{[L_m, L_n] = \delta_{m+n,0} \frac{(m-1)m(m+1)}{12} + (m-n) L_{m+n}}$$