Coordinate Bethe Ansatz for spin-1/2 fermions

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

We first consider a 2-particle problem to gain some intuitions

$$\psi_{\sigma_1\sigma_2}(x_1, x_2) = \sum_{QP} \theta(x_{Q_1} < x_{Q_2})\, A(Q, P)\, e^{i\sum_{i=1}^{2} k_{P_i} x_{Q_i}}$$

We will use P to represent permutation on momenta, and Q for presentation on positions.

$$\psi_{\sigma_1\sigma_2}(x_1, x_2) = \theta(x_1 < x_2)\left[A_{\sigma_1\sigma_2}(12,12)\, e^{i(k_1 x_1 + k_2 x_2)} + A_{\sigma_1\sigma_2}(12,21)\, e^{i(k_2 x_1 + k_1 x_2)}\right]$$
$$+\theta(x_2 < x_1)\left[A_{\sigma_1\sigma_2}(21,12)\, e^{i(k_1 x_2 + k_2 x_1)} + A_{\sigma_1\sigma_2}(21,21)\, e^{i(k_2 x_2 + k_1 x_1)}\right]$$

Fermi statistics $\psi_{\sigma_1\sigma_2}(x_1, x_2) = -\psi_{\sigma_2\sigma_1}(x_2, x_1)$

$$\psi_{\sigma_2\sigma_1}(x_2, x_1) = \theta(x_2 < x_1)\left[A_{\sigma_2\sigma_1}(12,12)\, e^{i(k_1 x_2 + k_2 x_1)} + A_{\sigma_2\sigma_1}(12,21)\, e^{i(k_2 x_2 + k_1 x_1)}\right]$$
$$+\theta(x_1 < x_2)\left[A_{\sigma_2\sigma_1}(21,12)\, e^{i(k_1 x_1 + k_2 x_2)} + A_{\sigma_2\sigma_1}(21,21)\, e^{i(k_2 x_1 + k_1 x_2)}\right]$$

$$\Rightarrow \boxed{\begin{aligned} A_{\sigma_1\sigma_2}(12,12) &= -A_{\sigma_2\sigma_1}(21,12) \\ A_{\sigma_1\sigma_2}(12,21) &= -A_{\sigma_2\sigma_1}(21,21) \end{aligned}}$$

Examples of $A_{\sigma_1\sigma_2}$, for a singlet state $\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)/\sqrt{2}$

$\rightarrow$ $A_{11} = 0$, $A_{1-1} = 1$, $A_{-1,1} = -1$, $A_{-1-1} = 0$.

it's eigenstate of $P_{\vec{\sigma}_1\vec{\sigma}_2} = \dfrac{1 + \vec{\sigma}_1 \vec{\sigma}_2}{2} =$

$$\begin{array}{cccc} |11\rangle & |1{-1}\rangle & |{-1}1\rangle & |{-1}{-1}\rangle \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

set $y = x_2 - x_1$, $X = \frac{x_1 + x_2}{2}$ $K = k_1 + k_2$, we have

$$\psi_{\sigma_1\sigma_2}(x_1, x_2) = e^{iKX} \begin{cases} A_{\sigma_1\sigma_2}(12,12)\, e^{i(k_2-k_1)y/2} + A_{\sigma_1\sigma_2}(12,21)\, e^{-i(k_2-k_1)y/2} \\ A_{\sigma_1\sigma_2}(21,12)\, e^{-i(k_2-k_1)y/2} + A_{\sigma_1\sigma_2}(21,21)\, e^{i(k_2-k_1)y/2} \end{cases}$$

① $\psi(y=0)$ is continuous

$$A_{\sigma_1\sigma_2}(12,12) + A_{\sigma_1\sigma_2}(12,21) = A_{\sigma_1\sigma_2}(21,12) + A_{\sigma_1\sigma_2}(21,21)$$

② $\left.\frac{\partial \psi}{\partial y}\right|_{0^+} - \left.\frac{\partial \psi}{\partial y}\right|_{0^-} = c\,\psi|_0$

$$\frac{i}{2}(k_2-k_1)\left[A_{\sigma_1\sigma_2}(12,12) - A_{\sigma_1\sigma_2}(12,21) + A_{\sigma_1\sigma_2}(21,12) - A_{\sigma_1\sigma_2}(21,21)\right]$$
$$= c\left[A_{\sigma_1\sigma_2}(12,12) + A_{\sigma_1\sigma_2}(12,21)\right]$$

Eqs 1, 2 are invariant under P_{12}, hence we can use P_{12} as a sym operator

define the exchange operator $T^Q_{Q_1Q_2}(Q_1 Q_2) = (Q_2, Q_1)$

$$T^Q_{Q_1Q_2} A_{\sigma_1\sigma_2}(Q_1 Q_2, P) = A_{\sigma_1\sigma_2}(Q_2 Q_1; P)$$

$$\Rightarrow \frac{i}{2}(k_2-k_1)\left[(1+\hat{T}^Q_{12})\left(A_{\sigma_1\sigma_2}(12,12) - A_{\sigma_1\sigma_2}(12,21)\right)\right]$$
$$= c\left[A_{\sigma_1\sigma_2}(12,12) + A_{\sigma_1\sigma_2}(12,21)\right]$$

$$\left[\frac{i}{2}(k_2-k_1)(1+\hat{T}^Q_{12}) - c\right] A_{\sigma_1\sigma_2}(12,12) = \left\{\frac{i}{2}(k_2-k_1)(1+\hat{T}^Q_{12}) + c\right\} A_{\sigma_1\sigma_2}(12,21)$$

multiply $\left[\frac{i}{2}(k_2-k_1)(-1+\hat{T}^Q_{12}) + c\right]$ to both side from the left

LHS:

$$\left(\frac{i}{2}(k_2-k_1)T^Q_{12} + c - \tfrac{i}{2}(k_2-k_1)\right)\left(\frac{i}{2}(k_2-k_1)T^Q_{12} - (c - \tfrac{i}{2}(k_2-k_1))\right)$$

$$= -\frac{(k_2-k_1)^2}{4} - \left(c - \tfrac{i}{2}(k_2-k_1)\right)^2 = -c^2 + ic\,(k_2-k_1)$$

RHS

$$\left[\frac{i}{2}(k_2-k_1)T_{12}^{Q}+c-\frac{i}{2}(k_2-k_1)\right]\left(\frac{i}{2}(k_2-k_1)T_{12}^{Q}+c+\frac{i}{2}(k_2-k_1)\right)$$

$$=\left(\frac{i}{2}(k_2-k_1)T_{12}^{Q}+c\right)^2+\frac{1}{4}(k_2-k_1)^2 = c^2+i(k_2-k_1)\hat{T}_{12}^{Q}$$

$$\Rightarrow \boxed{A_{\sigma_1\sigma_2}(12,12)=\frac{(k_1-k_2)\hat{T}_{12}^{Q}+iC}{(k_1-k_2)-iC}A_{\sigma_1\sigma_2}(12,21)}$$

This is a result based on both ① and ②, not just only from ②. but it has not counted Fermi statistics yet.

⊛ General solution for N-particles.

$$\psi_{\sigma_1\cdots\sigma_N}(x_1,\cdots x_N)=\sum_Q\sum_P \theta(x_{Q_1}<\cdots x_{Q_N})A_{\sigma_1\cdots\sigma_N}(Q,P)\cdot e^{i(k_{P_1}x_{Q_1}+\cdots k_{P_N}x_{Q_N})}$$

where $Q=(Q_1,\cdots Q_N)$, $P=(P_1,\cdots P_N)$

Consider two domains

$$Q=(Q_1,\cdots \boxed{Q_a=\xi,\ Q_b=\eta,}\cdots)$$

$$Q'=(Q'_1\cdots Q'_a,\ Q'_b\cdots)$$

$$=(Q_1,\cdots\boxed{Q_b=\eta,\ Q_a=\xi,}\cdots)$$

Q, Q' only differ by an exchange of neighboring elements.

Then in the domains Q and Q'

$$\boxed{\psi_{\cdot\cdot\sigma_\xi\cdots\sigma_\eta\cdot\cdot}(\cdots x_\xi\cdots x_\eta\cdots):}$$

$$:\ \theta(\cdots<x_{Q_a}<x_{Q_b}<\cdots)\sum_P A_{\cdot\cdot\sigma_\xi\cdots\sigma_\eta\cdot\cdot}(Q,P)\,e^{i\sum_{j=1}^{N}k_{P_j}x_{Q_j}}$$

$$+\ \theta(\cdots<x_{Q_b}<x_{Q_a}<\cdots)\sum_P A_{\cdot\cdot\sigma_\xi\cdots\sigma_\eta\cdot\cdot}(Q',P)\,e^{i\sum_{j=1}^{N}k_{P_j}x_{Q'_j}}$$

$$\boxed{\psi\left(\cdots x_\eta \cdots x_\xi \cdots\right)_{\cdots \sigma_\eta \cdots \sigma_\xi \cdots} = -\psi\left(\cdots x_\xi \cdots x_\eta \cdots\right)_{\cdots \sigma_\xi \cdots \sigma_\eta \cdots}}$$

$$\theta(\cdots x_{Q_b} < x_{Q_a} \cdots) \sum_P A_{\cdots \sigma_\eta \cdots \sigma_\xi \cdots}(Q, P)\, e^{i \sum k_{P_j} x_{Q'_j}}$$

$$+\,\theta(\cdots x_{Q_a} < x_{Q_b} \cdots) \sum_P A_{\cdots \sigma_\eta \cdots \sigma_\xi \cdots}(Q', P)\, e^{i \sum k_{P_j} x_{Q_j}}$$

$$\Rightarrow \boxed{A_{\cdots \sigma_\xi \cdots \sigma_\eta \cdots}(Q, P) = -A_{\cdots \sigma_\eta \cdots \sigma_\xi \cdots}(Q', P)} \leftarrow \text{Fermi statistics.}$$

let σ_ξ, σ_η run all the possible confIgs. it automatically covers

$$A_{\cdots \sigma_\eta \cdots \sigma_\xi \cdots}(Q, P) = -A_{\cdots \sigma_\xi \cdots \sigma_\eta \cdots}(Q', P). \leftarrow \text{redudant}$$

Now we calculate the scattering amplitude

$P = (\cdots P_a, P_b \cdots)$, $P' = (\cdots P_b\, P_a \cdots)$

$Q = (\cdots Q_a, Q_b \cdots)$, $Q' = (\cdots Q_b\, Q_a \cdots)$

① Given a, $b = a+1$ fixed (position)

② Arbitary $Q \to$ swith its Q_a and Q_b we arrive at Q'.

③ Arbitany $P \to$ swith its a, b th positins P'

$\Rightarrow$ Continuity Eq:

set $y = x_{Q_b} - x_{Q_a}$, $X = (x_{Q_b} + x_{Q_a})/2$, $K = k_{P_a} + k_{P_b}$

For the domain (sum half permutations over P)

$$\theta(\cdots x_{Q_a} < x_{Q_b} < \cdots) \sum_P{}' A_{\sigma_1 \cdots \sigma_N}(Q, P)\, e^{i(k_{P_1} x_{Q_1} + \cdots \underline{k_{P_a} x_{Q_a} + k_{P_b} x_{Q_b}} + \cdots k_{P_N} x_{Q_N})}$$
$$+ A_{\sigma_1 \cdots \sigma_N}(Q, P')\, e^{i(k_{P_1} x_{Q_1} + \cdots \underline{k_{P_b} x_{Q_a} + k_{P_a} x_{Q_b}} + \cdots k_{P_N} x_{Q_N})}$$

$$\theta(\cdots x_{Q_b} < x_{Q_a} < \cdots) \sum_P{}' A_{\sigma_1 \cdots \sigma_N}(Q', P)\, e^{i(k_{P_1} x_{Q_1} + \cdots \underline{k_{P_a} x_{Q_b} + k_{P_b} x_{Q_a}} + \cdots k_{P_N} x_{Q_N})}$$
$$+ A_{\sigma_1 \cdots \sigma_N}(Q', P')\, e^{i(k_{P_1} x_{Q_1} + \cdots \underline{k_{P_b} x_{Q_b} + k_{P_a} x_{Q_a}} + \cdots k_{P_N} x_{Q_N}}$$

① $\psi(y=0)$ contineous

$$A_{\sigma_1\cdots\sigma_N}(Q,P) + A_{\sigma_1\cdots\sigma_N}(Q,P') = A_{\sigma_1\cdots\sigma_N}(Q',P) + A_{\sigma_1\cdots\sigma_N}(Q',P')$$

② $$\partial\psi/\partial y\big|_{0^+} - \partial\psi/\partial y\big|_{0^-} = c\,\psi\big|_{y=0}.$$

$$\frac{i}{2}(k_{P_b}-k_{P_a})\left[A_{\sigma_1\cdots\sigma_N}(Q,P) - A_{\sigma_1\cdots\sigma_N}(Q,P') + A_{\sigma_1\cdots\sigma_N}(Q',P) - A_{\sigma_1\cdots\sigma_N}(Q',P')\right]$$
$$= c\left[A_{\sigma_1\cdots\sigma_N}(Q,P) + A_{\sigma_1\cdots\sigma_N}(Q,P')\right]$$

These two Eqs are symmetric with respect to $\hat{P}_{ab}$ ← the coordinate exchanges

$Q:(\cdots Q_a\, Q_b\cdots)$
ath bth positi[on]
$Q'=(\cdots Q_b\, Q_a\cdots)$

$$\Rightarrow \boxed{A_{\sigma_1\cdots\sigma_N}(Q,P) = \frac{k_{P_a}-k_{P_b}}{k_{P_a}-k_{P_b}-ic}A_{\sigma_1\cdots\sigma_N}(Q',P') + \frac{ic}{k_{P_a}-k_{P_b}-ic}A_{\sigma_1\cdots\sigma_N}(Q,P')}$$

define $$Y^{ab}_{P_bP_a} = \frac{(k_{P_a}-k_{P_b})\hat{P}^{Q}_{Q_aQ_b}+ic}{(k_{P_a}-k_{P_b})-ic} \rightarrow \boxed{A_{\sigma_1\cdots\sigma_N}(QP) = Y^{ab}_{P_bP_a}A_{\sigma_1\cdots\sigma_N}(Q,P')}$$

$$A_{\sigma_1\cdots\sigma_N}(\overset{Q}{Q_1\cdots \underline{Q_a}Q_b\cdots};\ \overset{P}{\cdots \underset{ath}{i}\underset{bth}{j}\cdots})$$

act on coordinate permutation
exchange the # at bth position

$$= \frac{(k_i-k_j)}{k_i-k_j-ic}A_{\sigma_1\cdots\sigma_N}(\underbrace{Q_1\cdots Q_bQ_a\cdots}_{Q'};\ \underbrace{P_1\cdots ji\cdots P_N}_{P'})$$
$$+\frac{ic}{k_i-k_j-ic}A_{\sigma_1\cdots\sigma_N}(\underbrace{Q_1\cdots Q_aQ_b\cdots}_{Q};\ \underbrace{P_1\cdots ji\cdots P_N}_{P'})$$

We often use the scattering matrix

$$A_{\cdots\sigma_{Qa}\cdots\sigma_{Qb}\cdots}(Q,P') = -A_{\cdots\sigma_{Qb}\cdots\sigma_{Qa}\cdots}(Q',P') \quad \leftarrow \text{Fermion statistics}$$

$$= -P_{\sigma_{Qa}\sigma_{Qb}} A_{\cdots\sigma_{Qa}\cdots\sigma_{Qb}\cdots}(Q',P')$$

where $\hat{P}_{\sigma_1\sigma_2} = \frac{1}{2}(1+\vec{\sigma}_1\cdot\vec{\sigma}_2) = \begin{cases} 1 & \text{triplet (symmetric in spin)} \\ -1 & \text{singlet (anti-sym in spin)} \end{cases}$

$$A_{\sigma_1\cdots\underline{\sigma_{Qa}\cdots\sigma_{Qb}}\cdots\sigma_N}(Q,P) = \frac{k_{P_b}-k_{P_a}}{k_{P_b}-k_{P_a}+iC} A_{\sigma_1\cdots\underline{\sigma_{Qa}\cdots\sigma_{Qb}}\cdots\sigma_N}(Q',P')$$

$$+ \frac{iC}{k_{P_b}-k_{P_a}+iC} A_{\sigma_1\cdots\underline{\sigma_{Qb}\cdots\sigma_{Qa}}\cdots\sigma_N}(Q',P')$$

$$\boxed{\begin{gathered} A_{\sigma_1\cdots\sigma_N}(Q,P) = S(k_{P_b}-k_{P_a}) A_{\sigma_1\cdots\sigma_N}(Q',P') \\ \text{where } S(k_{P_b}-k_{P_a}) = \frac{k_{P_b}-k_{P_a}+iC P_{\sigma_{Qa}\sigma_{Qb}}}{k_{P_b}-k_{P_a}+iC} \end{gathered}}$$

where $Q = (\cdots \underset{a\text{th}}{Q_a} \underset{b\text{th}}{Q_b} \cdots)$, $P = (\cdots \underset{a\text{th}}{P_a} \underset{b\text{th position}}{P_b} \cdots)$

$Q' = (\cdots Q_b Q_a \cdots)$ $\quad P' = (\cdots P_b P_a \cdots)$

if for SU(2) bosons

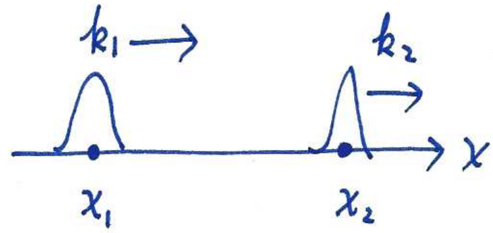
$$S(k_{P_b}-k_{P_b}) = \frac{k_{P_b}-k_{P_a} - iC\hat{P}_{\sigma_{Qa}\sigma_{Qb}}}{k_{P_b}-k_{P_a}+iC}$$

§ Scattering matrix,

$$\psi(x_1\sigma_1 ; x_2\sigma_2) = \Theta(x_1<x_2)\left[A_{\sigma_1\sigma_2}(12;12)\, e^{ik_1x_1+ik_2x_2} + A_{\sigma_1\sigma_2}(12;21)\, e^{ik_2x_1+ik_1x_2}\right]$$
$$+\,\Theta(x_2<x_1)\left[A_{\sigma_1\sigma_2}(21;12)\, e^{ik_1x_2+ik_2x_1} + A_{\sigma_1\sigma_2}(21;21)\, e^{ik_2x_2+ik_1x_1}\right]$$

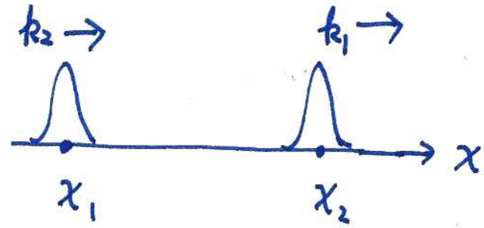
$A_{\sigma_1\sigma_2}(12;12)\, e^{ik_1x_1+ik_2x_2}$

incoming wave 1



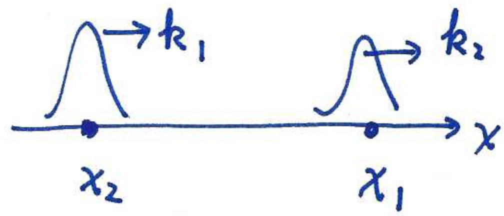
$A_{\sigma_1\sigma_2}(12;21)\, e^{ik_2x_1+ik_1x_2}$

out going wave 2



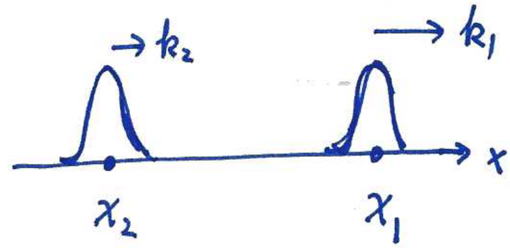
$A_{\sigma_1\sigma_2}(21;12)\, e^{ik_1x_2+k_2x_1}$

in coming wave 2



$A_{\sigma_1\sigma_2}(21;21)\, e^{ik_2x_2+ik_1x_1}$

out going wave 1



$$\boxed{A_{\sigma_1\sigma_2}(12;12) = S_{\sigma_1\sigma_2,\sigma_1'\sigma_2'}(k_2-k_1)\, A_{\sigma_1'\sigma_2'}(21;21)}$$

$$A_{\sigma_1\sigma_2}(21;12) = S_{\sigma_1\sigma_2,\sigma_1'\sigma_2'}(k_2-k_1)\, A_{\sigma_1'\sigma_2'}(12,21)$$

§ Yang - Baxter Eq

$$A_{\sigma_1 \cdots \sigma_N}(Q_1 \cdots Q_a Q_b \cdots ; P_1 \cdots P_a = i, P_b = j \cdots\cdots) =$$

$$= Y^{ab}_{ji} A_{\sigma_1 \cdots \sigma_N}(Q_1 \cdots Q_a Q_b \cdots ; P'_1, \cdots P'_a = j \; P'_b = i \cdots)$$

ath bth positions

$$\boxed{Y^{ab}_{ji} = (k_i - k_j) \, T^{Q}_{Q_a Q_b} + ic \Big/ [(k_i - k_j) - ic]}$$

use Y^{ab}_{ij} can change $A_{\sigma_1 \cdots \sigma_N}(Q, P = 12 \cdots N)$ to any $A_{\sigma_1 \cdots \sigma_N}(Q, P')$

but the methods can be more than one. We need to check the consistency conditions.

$$\boxed{① \quad Y^{ab}_{ij} Y^{ab}_{ji} = 1}$$

Proof: $\dfrac{(k_j - k_i) T^{Q}_{Q_a Q_b} + ic}{(k_j - k_i) - ic} \; \dfrac{(k_i - k_j) T^{Q}_{Q_a Q_b} + ic}{(k_i - k_j) - ic} = \dfrac{(ic)^2 - (k_i - k_j)^2}{(-ic)^2 - (k_i - k_j)^2} = 1$

$$Y^{ab}_{ij} Y^{ab}_{ji} A_{\sigma_1 \cdots \sigma_N}(Q_1 \cdots Q_a Q_b \cdots ; P_1 \cdots j \, i \cdots P_N)$$

ath bth

$$= Y^{ab}_{ij} A_{\sigma_1 \cdots \sigma_N}(Q_1 \cdots Q_a Q_b \cdots ; P_1 \cdots i \, j \cdots P_N)$$

$$= A_{\sigma_1 \cdots \sigma_N}(Q, P_1 \cdots j \, i \cdots P_N)$$

$$\boxed{② \quad Y^{ab}_{ij} Y^{cd}_{kl} = Y^{cd}_{kl} Y^{ab}_{ij} \quad \text{when } (ab), (cd) \text{ do not have common elements}}$$

$T_{Q_a Q_b}$, $T_{Q_c Q_d}$ then $(Q_a Q_b)$ and $(Q_c Q_d)$ commute $\Rightarrow Y^{ab}_{ij} Y^{cd}_{kl} = Y^{cd}_{kl} Y^{ab}_{ij}$

$$y^{ab}_{jk}\, y^{bc}_{ik}\, y^{ab}_{ij} = \frac{(k_k-k_j)T_{Q_aQ_b}+ic}{(k_k-k_j)-ic}\;\frac{(k_k-k_i)T_{Q_bQ_c}+ic}{(k_k-k_i)-ic}\;\frac{(k_j-k_i)T_{Q_aQ_b}+ic}{(k_j-k_i)-iC}$$

$$y^{bc}_{ij}\, y^{ab}_{ik}\, y^{bc}_{jk} = \frac{(k_j-k_i)T_{Q_bQ_c}+iC}{(k_j-k_i)-iC}\;\frac{(k_k-k_i)T_{Q_aQ_b}+ic}{(k_k-k_i)-ic}\;\frac{(k_k-k_j)T_{Q_bQ_c}+iC}{(k_k-k_j)-iC}$$

The denominates are the same.

- Check numerators

$$T_{Q_aQ_b}\,T_{Q_bQ_c}\,T_{Q_aQ_b} = \begin{pmatrix} Q_a & Q_b & Q_c \\ Q_b & Q_a & Q_c \end{pmatrix}\begin{pmatrix} Q_a & Q_b & Q_c \\ Q_a & Q_c & Q_b \end{pmatrix}\begin{pmatrix} Q_a & Q_b & Q_c \\ Q_b & Q_a & Q_c \end{pmatrix} = \begin{pmatrix} Q_a & Q_b & Q_c \\ Q_c & Q_b & Q_a \end{pmatrix}$$

$$T_{Q_bQ_c}\,T_{Q_aQ_b}\,T_{Q_bQ_c} = \begin{pmatrix} Q_a & Q_b & Q_c \\ Q_a & Q_c & Q_b \end{pmatrix}\begin{pmatrix} Q_a & Q_b & Q_c \\ Q_b & Q_a & Q_c \end{pmatrix}\begin{pmatrix} Q_a & Q_b & Q_c \\ Q_a & Q_c & Q_b \end{pmatrix} = \begin{pmatrix} Q_a & Q_b & Q_c \\ Q_c & Q_b & Q_a \end{pmatrix}$$

- Qudratic terms involes $T' \cdot T$ are identical.

- linear terms $\left\{ T_{Q_aQ_b}(k_k-k_j+k_j-k_i) + T_{Q_bQ_c}(k_k-k_i) \right\} (ic)^2$

$$= \left\{ T_{Q_aQ_b}(k_k-k_i) + T_{Q_bQ_c}(k_j-k_i+k_k-k_j) \right\} (ic)^2$$

/

index of position

a b c

$$\begin{tikzcd}
A_{\sigma_1\cdots\sigma_N}(Q,\cdots ijk\cdots) \arrow[r, "y^{ab}_{ij}"] \arrow[d, "y^{bc}_{jk}"] & A_{\sigma_1\cdots\sigma_N}(Q,\cdots jik) \arrow[r, "y^{bc}_{ik}"] & A_{\sigma_1\cdots\sigma_N}(Q,\cdots jki\cdots) \arrow[d, "y^{ab}_{jk}"] \\
A_{\sigma_1\cdots\sigma_N}(Q,\cdots ikj) \arrow[r, "y^{ab}_{ik}"] & A_{\sigma_1\cdots\sigma_N}(Q,\cdots kij\cdots) \arrow[r, "y^{bc}_{ij}"] & A_{\sigma_1\cdots\sigma_N}(Q,\cdots kji\cdots)
\end{tikzcd}$$

index of momentum

$$y^{ab}_{jk}\, y^{bc}_{ik}\, y^{ab}_{ij} = y^{bc}_{ij}\, y^{ab}_{ik}\, y^{bc}_{jk}$$

Check periodical boundary condition

$$\psi(x_1\sigma_1, \cdots x_N\sigma_N) = \sum_{Q} \theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N}) \sum_{P} A_{\sigma_1 \cdots \sigma_N}(Q.P)\, e^{i(k_{P_1} x_{Q_1} + \cdots k_{P_N} x_{Q_N})}$$

Consider the domain that $x_1 < x_{Q_2} < \cdots x_{Q_N}$, i.e. $Q_1 = 1$,

then $x_{Q_2} < \cdots < x_{Q_N} < x_1 + L \Rightarrow \psi(x_1\sigma_1, \cdots x_N\sigma_N) = \psi(x_1+L, \sigma_1, \cdots x_N\sigma_N)$

in $\psi(x_1+L\,\sigma_1, \cdots x_N\sigma_N)$, we have $\theta(x_{Q_2} < \cdots x_{Q_N} < x_{Q_1}+L) \sum_P A_{\sigma_1 \cdots \sigma_N}(Q', P)\, e^{i(k_{P_1} x_{Q_2} + \cdots + k_{P_N}(x_1+L))}$

change $P \to P' = (P_2, P_3 \cdots P_N, P_1)$

$$\Rightarrow \theta(x_{Q_2} < \cdots < x_{Q_N} < x_1 + L) \sum_P A_{\sigma_1 \cdots \sigma_N}(Q_2 Q_3 \cdots Q_N Q_1, P_2 \cdots P_N P_1)\, e^{i(k_{P_2} x_{Q_2} + \cdots k_{P_N} x_{Q_N} + k_{P_1}(x_1+L))}$$

Compare phase factor . check the term $e^{i(k_{P_1} x_{Q_1} + \cdots k_{P_N} x_{Q_N})}$

$$\Rightarrow \boxed{A_{\sigma_1 \cdots \sigma_N}(Q_1 \cdots Q_N; P_1 \cdots P_N) = A_{\sigma_1 \cdots \sigma_N}(Q_2 \cdots Q_N Q_1; P_2 \cdots P_N P_1)\, e^{i k_{P_1} L}}$$

plug in the case of

$$(Q_1 \cdots Q_N) = (P_1 \cdots P_N) = (j\ 1 \cdots j-1, j+1, \cdots N) \to (Q_2 \cdots Q_N Q_1) = (1 \cdots j-1\ j+1, \cdots N\, j)$$

$$A_{\sigma_1 \cdots \sigma_N}(j\ 1 \cdots j-1, j+1, \cdots N;\ j\ 1 \cdots j-1, j+1 \cdots N)$$

$$= e^{i k_j L} A_{\sigma_1 \cdots \sigma_N}(1 \cdots j-1, j+1 \cdots N\, j; 1 \cdots j-1, j+1, \cdots N, j)$$

Consider $P = (\cdots P_a P_b \cdots)$ $\quad P' = (\cdots P_b P_a \cdots)$

$Q = (\cdots Q_a Q_b \cdots)$ $\quad Q' = (\cdots Q_b Q_a \cdots)$

Using $A_{\sigma_1 \cdots \sigma_N}(Q, P) = S(k_{P_b} - k_{P_a})\, A_{\sigma_1 \cdots \sigma_N}(Q', P')$

where $$S(k_{P_b} - k_{P_a}) = \frac{k_{P_b} - k_{P_a} + ic\, P_{\sigma_{Q_a}\sigma_{Q_b}}}{k_{P_b} - k_{P_a} + ic}$$

$\longrightarrow$ simplify $$S_{ij} = \frac{k_i - k_j + ic\, P_{\sigma_i \sigma_j}}{k_i - k_j + ic}$$ set $P_a = Q_a = i$, $P_b = Q_b = j$

$\Rightarrow$ $$A_{\sigma_1 \cdots \sigma_N}(j\,1\,2 \cdot j{-}1, j{+}1 \cdots N;\ j\,1\,2 \cdot j{-}1\, j{+}1 \cdots N)$$

$$= S_{1j}\, S_{2j} \cdots S_{j-2,j}\, S_{j-1,j}\, A_{\sigma_1 \cdots \sigma_N}(1\,2 \cdots N,\ 1\,2 \cdots N)$$

(move j all the way to the very left position).

Similarly $$A_{\sigma_1 \cdots \sigma_N}(1 \cdots j{-}1\ j{+}1 \cdot N\, j;\ 1 \cdots j{-}1, j{+}1 \cdots N\, j)$$

$$= S_{j,N} \cdots S_{j\,j+2}\, S_{j\,j+1}\, A_{\sigma_1 \cdots \sigma_N}(1\,2 \cdots N,\ 1\,2, \cdots N)$$

move j all the way to the very right position.

$\Rightarrow$ $$\cancel{[S_{j,N} \cdots S_{j,j+1}]}^{-1} (S_{1j}\, S_{2j} \cdots S_{j-1,j})\, A(1\,2 \cdots N;\ 1\,2 \cdots N)$$

$$= (S_{j,N} \cdots S_{j\,j+1})\, e^{ik_j L}\, A(1\,2 \cdots N;\ 1\,2 \cdots N)$$

$$\Rightarrow \boxed{\begin{aligned} &\left[S_{j+1,j}\, S_{j+2,j} \cdots S_{N,j}\ S_{1j}\, S_{2j} \cdots S_{j-1\,j}\right] A_{\sigma_1 \cdots \sigma_N}(1\,2 \cdots,\ 1\,2 \cdots N) \\ &= e^{ik_j L}\, A_{\sigma_1 \cdots \sigma_N}(1\,2 \cdots N;\ 1\,2 \cdots N) \end{aligned}}$$

We need to solve the eigenvalue of

$$S_{j+1,j} \cdots S_{Nj} \; S_{1j} \cdots S_{j-1,j} \longrightarrow e^{ik_jL}$$

And $A_{\sigma_1 \cdots \sigma_N}$ as its eigenvector

$$\Rightarrow E = \sum_{j=1}^{N} k_j^2$$

with $$S_{ij} = \frac{k_i - k_j + icP_{\sigma_i \sigma_j}}{k_i - k_j + ic}$$