# Bethe Ansatz (I) - Fundamentals

## Contents

# 1 Why Bethe Ansatz?

Figure 1: Elastic collision of two balls in one dimension. The momenta $k_1$ and $k_2$ switch.

Bethe Ansatz (BA) is a powerful method to solve a class of problems called integrable models. It comes from very physical intuitions from classic collisions. Consider two balls in one dimension with momenta $k_1$ and $k_2$. For the elastic collision, both momentum and kinetic energy are conserved. Hence, the balls just switch their momenta, i.e., not only $k_1 + k_2$ is conserved, but also each of them is conserved. This property can be generalized to $N$ balls, in which each of the momenta $k_1, k_2, ...., k_N$ is conserved, and they just exchange after each collision. Since there are $N$ conserved quantities, such a system is integrable classically. This important feature does not work in two-dimensions and above. Basing on this observation, Bethe proposed the quantum mechanical wavefunction for quantum particles in 1D – the Bethe Ansatz. Later on, it becomes an entire branch of mathematical physics – integrable models. Why this direction is important?

1. Beautiful mathematical structure – integrable models, quantum groups
2. All the spectra, not just the low energy sector, beyond effective field theory
3. Comparable to experiment /Nature 554, 219 (2018)
4. Application to string theory
5. Calibration to numerical method, and field theory method
6. New excitations: spionons, psinons, magnons

# 2 Spin-$\frac{1}{2}$ Heisenberg model

Quantum Heisenberg model is a basic model to describe quantum magnetism. In fact, they are consequences of the electric interaction when combined with Pauli's exclusion principle, rather than the magnetic dipolar interaction which is typically too small in solids. In solid state classes, you should learn that they are called exchange interaction for the ferromagnetic (FM) case, and superexchange interaction for the anti-ferromagnetic (AFM) case.

Consider the spin-1/2 Heisenberg spin chain (one-dimension)

$$H = \frac{J}{2}\sum_{x=1}^{N}\Big\{S_x^+ S_{x+1}^- + S_i^- S_{x+1}^+ + 2\Delta(S_x^z S_{x+1}^z - \frac{1}{4})\Big\}, \tag{1}$$

where $S_x^{\pm} = S_x^x \pm i S_x^y$, and the subindex $x$ is the site index. We use the periodical boundary condition for spin $\vec{S}_{x+N} = \vec{S}_x$. Eq. 1 has an $U(1)$ symmetry, i.e., the total $S^z = \sum_x S_x^z$ is conserved.

We start with the reference state $|\Psi_0\rangle = |\uparrow ...... \uparrow\rangle$ that every spin is polarized as upward. Based on $|\Psi_0\rangle$, we flip $M$ spin ups to downs, and $M$ is the number of magnons, and then

$$S_z = \frac{N}{2} - M. \tag{2}$$

Then we represent

$$|\Psi_M\rangle = \sum_{x_1<x_2<...<x_M} \psi(x_1, x_2, ..., x_N) S^-_{x_1} S^-_{x_2} ... S^-_{x_N} |\Psi_0\rangle, \tag{3}$$

where $\psi(x_1, x_2, ..., x_N)$ can be viewed as wavefunctions in the coordinate representation.

Next we set up the Schrödinger equaiton: $H\Psi|\rangle = E\Psi|\rangle$. We have

$$\begin{aligned}
\sum_x S^+_x S^-_{x+1}|\Psi\rangle &= \sum_{x_1<x_2<...<x_M} \Big\{ \sum_{l=1}^{M} \psi(x_1, .., x_{l-1}, x_l - 1, x_{l+1}, ..., x_M \Big\} |x_1, ...x_M\rangle \\
\sum_x S^-_x S^+_{x+1}|\Psi\rangle &= \sum_{x_1<x_2<...<x_M} \Big\{ \sum_{l=1}^{M} \psi(x_1, .., x_{l-1}, x_l + 1, x_{l+1}, ..., x_M \Big\} |x_1, ...x_M\rangle. \quad (4)
\end{aligned}$$

In the above expressions, we have use the convention that if $x_l + 1 = x_{l+1}$, or, $x_l - 1 = x_{l-1}$, then the corresponding term will not be counted. We also have

$$2\Delta\Big\{\sum_x S^z_x S^z_{x+1} - \frac{1}{4}\Big\}|\Psi\rangle = -\Delta \sum_{x_1<x_2<...<x_M} n(x_1, x_2, ..., x_M)\psi(x_1, ..., x_M)|\Psi\rangle, \tag{5}$$

where $n(x_1, ..., x_M)$ is the number of domains between $\uparrow$ and $\downarrow$, i.e., ... $\uparrow\downarrow$ ...., and .... $\downarrow\uparrow$ .... Then from $H|\Psi\rangle = E|\Psi\rangle$, we arrive at the following equations

$$\begin{aligned}
&\frac{J}{2}\sum_{l=1}^{M}\Big\{\psi(x_1, ..., x_{l-1}, x_l + 1, x_{l+1}, ..., x_M) + \psi(x_1, ..., x_{l-1}, x_l - 1, x_{l+1}, ..., x_M)\Big\} \\
-\;&\frac{J}{2}\Delta n(x_1, ..., x_M)\psi(x_1, ..., x_M) = E\psi(x_1, ..., x_M), \qquad (6)
\end{aligned}$$

under the convention that if $x_l$ and $x_{l+1}$, or, $x_{l-1}$ and $x_l$ are neighbors, then the corresponding $\psi(x_1, ..., x_{l-1}, x_l + 1, x_{l+1}, ..., x_M)$ and $\psi(x_1, ..., x_{l-1}, x_l - 1, x_{l+1}, ..., x_M)$ do not exist.

**Exercise**

Prove the above expressions Eqs. 4 and 5, 6.

**Exercise**

Now let us figure some simple states. The zero magnon state is simply the vacuum state, or the reference state $|\Psi_0\rangle = |\uparrow\uparrow .... \uparrow\rangle$ As for the single magnon state, i.e., $M = 1$, there is no interaction.

1) Work out the dispersion of the single magnon state for the FM spin chain,

$$E(k) = |J|(\Delta - \cos k), \tag{7}$$

with $k = n\frac{2\pi}{N}$ and $n = 0, 1, ..., N-1$.

2) Now consider the case of SU(2) case, i.e., the isotropic Heisenberg model with $\Delta = 1$, show that magnon excitations are gapless. Figure out the good quantum numbers of the total spin $S$ and $S_z$ for each of the $N$ one-magnon state.

# 3 The BA wavefunction

For a classic collision problem, at each time, there is only one permutation of momentum among different balls. But for a quantum mechanical problem, we need to superpose plane waves with all the possible permutations – this is the Bethe Ansatz wavefunction defined as follows

$$\psi(x_1, x_2, ..., x_M) = \sum_P A_p e^{i\sum_{i=1}^{M} k_{Pl} x_l}, \tag{8}$$

where $P$ is a permutation $(P1, P2, ...., PN)$; $A_P$ is the scattering amplitude of each scattering state. The interaction effect is contained in the ratios of $A_{P'}/A_P$ represents the many-body scattering amplitudes between configurations $P'$ and $P$.

There are two problems to be solved:

1. What are the relations among $A_P$'s?
2. What are the requirement for the values of $k_i$?

The first one is reduced to solve a 1D two-body scattering problem, and it turns out that the many-body scattering amplitude can be factorized into a product of a sequence of two-body scattering amplitudes. It is based on such a fact that a general permutation can be expressed a product of two-body exchanges. And this is the essence of integrable problem.

Remark: We can view magnons as hard-core bosons. If we work in the 1st quantization picture, the wavefunction should be written as

$$\begin{aligned} \Phi(x_1, x_2, ..., x_N) &= \theta(x_1 < x_2 < ... < x_N)\psi(x_1, x_2, ..., x_N) \\ &+ .... \\ &+ \theta(x_{P1} < x_{P2} < ... < x_{PN})\psi(x_{P1}, x_{P2}, ..., x_{PN}) \\ &+ .... \end{aligned} \tag{9}$$

Such a wavefunction satisfies the permutation symmetry of a bosonic wavefunction. In other words, $\psi(x_1, x_2, ...x_N)$ is the wavefunction only for a domain of $x_1 < x_2 < ... < x_N)$. Nevertheless, once the wavefunction in this domain is known, it is completely determined in all other domains.

**Exercise**

1. Prove that the eigen energy of the Bethe Ansatz wavefunction Eq. 8 is

$$E = J\sum_{j=1}^{M}(\cos k_j - \Delta). \tag{10}$$

2. The periodical boundary condition. If we want to apply $\Phi(x_1, x_2, ..., x_N) = \Phi(x_1 + L, x_2, ..., x_N) = \Phi(x_1, x_2 - L, ..., x_N)$, they give rise to

$$\psi(x_1, x_2, ..., x_N) = \psi(x_2, x_3, ..., x_N, x_1 + L) = \psi(x_2 - L, x_1, ...., x_N) \tag{11}$$

# 4 Two magnon states – scattering

We write down the two-magnon wavefunction

$$\psi(x_1, x_2) = A(12)e^{ik_1x_1 + ik_2x_2} + A(21)e^{ik_2x_1 + ik_1x_2}, \tag{12}$$

where $x_1 < x_2$. If $x_1$ and $x_2$ are not adjacent, then the above wavefunction satisfies

$$\frac{J}{2}\Big(\psi(x_1 + 1, x_2) + \psi(x_1 - 1, x_2) + \psi(x_1, x_2 - 1) + \psi(x_1, x_2 + 1)\Big) = E\psi(x_1, x_2) \tag{13}$$

Now consider the case that they are adjacent, i.e., $x_1 + 1 = x_2$, then we have

$$\frac{J}{2}\Big(\psi(x_1 - 1, x_2) + \psi(x_1, x_2 + 1)\Big) - J\Delta\psi(x_1, x_2) = 0 \tag{14}$$

Comparing with Eq. 13 and Eq. 14, we can take

$$\frac{J}{2}\Big(\psi(x_1 + 1, x_2) + \psi(x_1, x_2 - 1)\Big) - J\Delta\psi(x_1, x_2) = 0. \tag{15}$$

Then if we want to use Eq. 13 to describe the wavefunction for all the cases, we can set the boundary condition as

$$\frac{J}{2}\Big(\psi(x_2, x_2) + \psi(x_1, x_1)\Big) = J\Delta\psi(x_1, x_2), \tag{16}$$

by setting $x_2 = x_1 + 1$.
In the following, we use the convention that

$$\frac{A'}{A} = \frac{A(21)}{A(12)} = -e^{i\Theta(k_2, k_1)}. \tag{17}$$

**Exercise**
1) Plugging the BA wavefunction Eq. 12 into the boundary condition Eq 16, derive that scattering amplitude

$$e^{i\Theta(k_1, k_2)} = \frac{e^{i(k_1 + k_2)} - 2\Delta e^{ik_1} + 1}{e^{i(k_1 + k_2)} - 2\Delta e^{ik_2} + 1}. \tag{18}$$

2) For the isotropic case, i.e. $\Delta = 1$, we parameterize

$$e^{ik_i} = \frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}, \tag{19}$$

then it means that $\lambda_i = \frac{1}{2}\cot\frac{k_i}{2}$ following the illustration in Fig. 2.

3) Prove that at $\Delta = 1$

$$e^{i\Theta(k_1,k_2)} = -\frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}. \qquad (20)$$

Figure 2: Rapidity

# 5 The BA equation

We have derived that the periodical boundary conditions

$$\psi(x_1, x_2) = \psi(x_2, x_1 + N) = \psi(x_2 - N, x_1) = \psi(x_1 + N, x_2 + N). \qquad (21)$$

**Exercise**
Prove that the above periodical boundary conditions give rise to

$$\begin{aligned} e^{ik_1N} = \frac{A(12)}{A(21)} &= (-)e^{-i\Theta(k_2,k_1)} = (-)e^{i\Theta(k_1,k_2)}, \\ e^{ik_2N} = \frac{A(21)}{A(12)} &= (-)e^{-i\Theta(k_1,k_2)} = (-)e^{i\Theta(k_2,k_1)}. \end{aligned} \qquad (22)$$

Then the above equations are called the Bethe ansatz equations. From the equations above, we have

$$e^{i(k_1+k_2)N} = 1. \qquad (23)$$

Then how to understand the above results? Let us view particles 1 and 2 as indistinguishable bosons, and use the extended wavefunction $\Phi(x_1, x_2)$ which covers both domains that $x_1 < x_2$ and $x_2 < x_1$, i.e.,

$$\begin{aligned} \Phi(x_1, x_2) &= \theta(x_1 < x_2)\Big(A(12)e^{ik_1x_1+ik_2x_2} + A(21)e^{ik_2x_1+ik_1x_2}\Big) \\ &+ \theta(x_2 < x_1)\Big(A(12)e^{ik_1x_2+ik_2x_1} + A(21)e^{ik_2x_2+ik_1x_1}\Big). \end{aligned} \qquad (24)$$

Let us elaborate what happens when we move particle 1 from $x_1$ to $x_1+N$ during which particle 2 is fixed at $x_2$, and initially $x_1 < x_2$ as shown in Let us trace the $A(12)$ term in the domain $\theta(x_1 < x_2)$, in which particle 1 carries momentum $k_1$.

Figure 3: Understanding the periodical boundary condition

1. As moving particle 1 from $x_1$ to $x_2 - 1$, particle 1 acquires the phase $e^{ik_1(x_2-x_1-1)}$.

2. Particle 1 hops from $x_2 - 1$ to $x_2 + 1$, then the system enters the domain of $\theta(x_2 < x_1)$. To trace the particle $x_1$ which still has the momentum $k_1$, we need to look at the $A(21)$ term in the domain of $\theta(x_2 < x_1)$. In this process, particle 1 gains the phase of $A(21)/A(12)e^{ik_12}$.

3. The last step, we move particle 1 to $x_1+N$, and it gains the phase of $e^{ik_1(x_1+N-x_2-1)}$.

Combine the phases gained in the above three steps, we have

$$\frac{A(21)}{A(12)} e^{ik_1 N} = 1. \tag{25}$$

In other words, interaction shifts the ordinary periodical boundary condition by taking into account the additional phase shift from the 2-body scattering. This changes the quantization rule of $k_1$ from the free case. Similarly, we can also trace the configuration in the $A(21)$-term starting in the domain of $\theta(x_1 < x_2$, which represents that particle 1 carries the momentum $k_2$, and particle 2 carries the momentum $k_1$. Repeating the above analysis, we arrive at

$$\frac{A(12)}{A(21)} e^{ik_2 N} = 1. \tag{26}$$

Certainly, if we shift $x_1 \to x_1 + N$ and $x_2 \to x_2 + N$ while keeping their distance unchanged, there are no additional scattering phase since they stay in the same domain. We have

$$e^{i(k_1+k_2)N} = 1. \tag{27}$$

**Exercise**:
For the isotropic case of $\Delta = 1$, plug in the parameterization of the rapidity $\lambda_i$, Prove that

$$\begin{aligned} \Big(\frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}}\Big)^N &= \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}, \\ \Big(\frac{\lambda_2 + \frac{i}{2}}{\lambda_2 - \frac{i}{2}}\Big)^N &= \frac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i}. \end{aligned} \tag{28}$$

# 6 The existence of bound state

Bound states correspond to complex solutions of $k$, hence, as $N \to \infty$, we have $\Big((\lambda_1 + i/2)/(\lambda_1 - i/2)\Big)^N \to 0$ or $\infty$, which means that $\lambda_1 - \lambda_2 = \pm i$.

Then the total energy

$$\begin{aligned} E/J &= \cos k_1 + \cos k_2 - 2 = \frac{1}{2}(e^{ik_1} + 1/e^{ik_1} + e^{ik_2} + 1/e^{ik_2} - 2) \\ &= \frac{1}{2}\Big(\frac{\lambda_1 + i/2}{\lambda_1 - i/2} + \frac{\lambda_1 - i/2}{\lambda_1 + i/2} + \frac{\lambda_2 + i/2}{\lambda_2 - i/2} + \frac{\lambda_2 - i/2}{\lambda_2 + i/2} - 4\Big) \\ &= J\big(\frac{\lambda_1^2 - 1/4}{\lambda_1^2 + 1/4}\frac{\lambda_2^2 - 1/4}{\lambda_2^2 + 1/4} - 2\big) \\ &= -\frac{J}{2}\Big(\frac{1}{\lambda_1^2 + 1/4} + \frac{1}{\lambda_2^2 + 1/4}\Big). \end{aligned} \tag{29}$$

The above derivation remains valid even $k_i$ is complex.

Set $\lambda_{1,2} = x \pm i/2$, we have the bound state energy

$$E_b = -\frac{J}{2}\frac{1}{x^2+1}. \tag{30}$$

In order $E$ to be real, we need $x$ to be either real, or, purely imaginary. If $x$ is purely imaginary, then $\lambda_{1,2}$ are imaginary, and so do $k_{1,2}$, which cannot be the case, hence $x$ is real. In this case, we have

$$e^{ik_1+ik_2} = \frac{x+i}{x-i}, \tag{31}$$

hence, $\cos(k_1+k_2) = (x^2-1)/(x^2+1)$. We have

$$E_b = \frac{J}{2}\Big(\cos(k_1+k_2) - 1\Big), \tag{32}$$

For the real values of $k_{1,2}$, it can be proved that

$$\frac{1-\cos(k_1+k_2)}{2} \leq 1 - \cos k_1 + 1 - \cos k_2. \tag{33}$$

Hence, for the FM case, $J < 0$, we have the

Figure 4: Understanding the periodical boundary condition

$$\begin{aligned} E_b &= |J|(1-\cos(k_1+k_2) \\ E_{scattering} &= |J|(1-\cos k_1 + 1 - \cos k_2) = 2|J|(1-\cos\frac{k_1+k_2}{2}\cos\frac{k_1-k_2}{2}). \end{aligned}$$

The upper and lower boundaries of the scattering states of the FM case are

$$E_\pm(k) = 2|J|(1 \pm \cos\frac{k}{2}), \tag{34}$$

where $k$ is the center of mass momentum. The bound state energy lies out of the scattering state continuum as shown in Fig. 4. If for the AFM case, the bound state is at a higher energy.

For many-body version of the bound state, string states, in the AFM system, please refer to


1. W. Yang et al, arxiv 1702.01854,

2. Z. Wang et al, Nature 554, 219 (2018)


**Homework:**
**Please perform numerical solutions for the BA equation to all the 2-magnon states for a chain of AFM spin-1/2 Heisenberg model with $\Delta = 1$ and with a finite length, say, $N = 10$. Find the momenta of magnons and the eigen energies. Pay attention to the bound states.**