Lect 1 Bethe Ansatz (I) — Fundamentals

① Why Bethe Ansatz ?

a: beautiful mathematical structure — integrable, quantum groups.

b: all the spectra, not just the low energy — beyond field theory

c: comparable to experiment, / Nature 554, 219 (2018)
arxiv 1702.01854

d: Application to string theory.

e: Calibration to numerical method, and field theory method

f: new excitations: spinons, psinons, magnons, etc.

② Heisenberg spin chain - spin-1/2

$$H = \frac{J}{2} \sum_{x=1}^{N} \{ S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+ + 2\Delta [S_x^3 S_{x+1}^3 - 1/4] \}$$

x: index of lattice sites

$$\vec{S}_x : \; 1 \otimes 1 \cdots \otimes \underset{\substack{\uparrow \\ x\text{-site}}}{\frac{\vec{\sigma}}{2}} \otimes 1 \cdots \otimes 1$$

$$S_x^{\pm} = S_x^1 \pm i S_x^2$$

periodical boundary condition $\vec{S}_{x+N} = \vec{S}_x$

total S^3 conserved, $[\sum_x S^3_x, H] = 0$.

For states within the sector of $\sum_x S^3_x = \frac{N}{2} - M$, we call it

M-particle states,

$$|\psi\rangle = \sum_{x_1 < \cdots x_M} \psi(x_1, \cdots x_M)\, S^-_{x_1} \cdots S^-_{x_M} |\underset{1}{\uparrow} \cdots \underset{N}{\uparrow}\rangle$$

$$= \sum_{x_1 < \cdots < x_M} \psi(x_1, \cdots x_M)\, |x_1, \cdots x_M\rangle$$

Note: The fully polarized state $|\uparrow \cdots \uparrow\rangle$ is viewed as the vacuum state, and the flipped spins are viewed as particles.

Setting up the Schrödinger Eq: $H|\psi\rangle = E|\psi\rangle$.

① $$\sum_x S^+_x S^-_{x+1} |\psi\rangle = \sum_{x_1 < \cdots < x_M} \left\{ \sum_{l=1}^{M} \psi(x_1, \cdots x_{l-1}, \boxed{x_l - 1}, x_{l+1}, \cdots x_M) \right\} |x_1, \cdots \cdots x_M\rangle$$

The convention is, if $x_{l-1} = x_l - 1$, then $\psi(x_1, \cdots x_{l-1}, x_l - 1, \cdots x_M)$ will not be counted in the summation. This process describes the hopping process of flipped spins.

$$\underset{x_l}{\downarrow}\ \underset{x_l+1}{\uparrow} \xrightarrow{S^+S^-} \uparrow\ \downarrow, \qquad \underset{x_{l-1}}{\downarrow}\ \underset{x_l}{\downarrow}\ \underset{x_l+1}{\uparrow} \xrightarrow{\sum S^+S^-} \downarrow\ \uparrow\ \downarrow$$

② $\left(\sum_x S_x^- S_{x+1}^+\right)|\psi\rangle = \sum_{x_1<\cdots x_M} \left\{ \sum_{l=1}^{M} \psi(x_1,\cdots x_{l-1},\boxed{x_l+1},x_{l+1},\cdots x_M)\right\}|x_1,\cdots x_M\rangle$

if $x_l+1 = x_{l+1}$, then $\psi(x_1\cdots x_{l-1}, x_l+1, x_{l+1},\cdots x_M)$ will not be counted.

$$\uparrow \ \underset{x_l}{\downarrow} \ \underset{x_{l+1}}{\downarrow} \ \uparrow \xrightarrow{\Sigma S^- S^+} \downarrow \ \uparrow \ \downarrow \ \uparrow$$

③ $2\Delta\left(\sum_x S_x^3 S_x^3 - 1/4\right)|\psi\rangle = -\Delta \sum_{x_1<\cdots x_M} n(x_1,\cdots x_M)\,\psi(x_1\cdots x_M)\,|x_1,\cdots x_M\rangle$

where $n(x_1,\cdots x_M)$ is the # of domain $\cdots\uparrow\downarrow\cdots$, and $\cdots\downarrow\uparrow\cdots$, or the # of broken bonds.

Proof: we use ① as an example

$$\sum_x S_x^+ S_{x+1}^- \sum_{x_1<\cdots x_M} \psi(x_1,\cdots x_M)\,|x_1,\cdots x_M\rangle$$

$$= \sum_{x_1<\cdots x_M} \psi(x_1,\cdots x_l\cdots x_M) \sum_{l=1}^{M} |x_1,\cdots x_{l-1}, x_l+1, x_{l+1},\cdots x_M\rangle$$

$$= \sum_{x_1<\cdots x_M}\left[\left\{\sum_{l=1}^{M} \psi(x_1,\cdots x_{l-1}, x_l-1, x_{l+1},\cdots x_M)\right\}|x_1,\cdots x_M\rangle\right\}$$

For ③ each parallel config gives rise $S_x^3 S_{x+1}^3 - 1/4 = 0$,

For anti-paralleed bond $S_x^3 S_{x+1}^3 - 1/4 = -1/2$.

Then from $H|\psi\rangle = E|\psi\rangle$, compare the coefficients of the basis of $|x_1, \cdots x_M\rangle$

$\Rightarrow$

$$(*) \quad \frac{J}{2}\sum_{l=1}^{M} \psi(x_1,\cdots x_{l-1}, \underline{x_l+1}, x_{l+1},\cdots x_M) + \psi(x_1,\cdots x_{l-1}, \underline{x_l-1}, x_{l+1},\cdots x_M)]$$

$$-\frac{J}{2}\,\Delta n(x_1,\cdots x_M)\,\psi(x_1,\cdots x_M) = E\,\psi(x_1,\cdots x_M)$$

if x_l, x_{l+1}, or x_{l-1}, x_l are neighbors, then the corresponding $\psi(x_1\cdots x_{l-1}, x_l+1, x_{l+1},\cdots x_M)$ and $\psi(x_1,\cdots x_{l-1}, x_l-1, \cdots x_M)$ don't exist.

★Hint from classic collisions in 1D

Not only k_1+k_2 conserve, but also k_1, k_2 are seperately conserved.

but they permute. → N balls, there're n-conserved quantities → integrable.

In QM, we need to superpose all the configurations together and calculate scattering amplitude.

$$\psi(x_1,\cdots x_M) = \sum_P A_P \, e^{i\sum_{l=1}^{M} k_{P_l} x_l},$$ where P is a permutation $P=(P_1,\cdots P_M)$

Bethe ansatz → each magon is in the plane wave

* Remark: magons are also hard-core bosons, but $\psi(x_1, \cdots x_M)$ does not need to satisfy the boson permutation symmetry. The reason is the sequence $1 \le x_1 < x_2 < \cdots x_M \le N$, it corresponds to a particular a range, the sequence doesn't change as motion. In other words, we can view it as the boson WF in the region $1 \le x_1 < x_2 < \cdots \le N$. We do not introduce particle index here.

- Energy of Bethe states

Where all the "↓" are not neighbours, $n(x_1, \cdots x_M) = 2M$

Plug in the Bethe WF,

$$\psi(x_1, \cdots x_\ell+1, \cdots x_M) + \psi(x_1, \cdots x_\ell - 1, \cdots x_M)$$

$$= \sum_P A_P \, e^{i \sum_{\ell'=1}^{M} k_{P\ell'} x_{\ell'}} \left[e^{i k_{P\ell}} + e^{-i k_{P\ell}} \right] = \sum_P 2 A_P \cos k_{P\ell} \, e^{i \sum_{\ell'=1}^{M} k_{P\ell'} x_{\ell'}}$$

$$\Rightarrow \sum_{\ell=1}^{M} \psi(x_1, \cdots x_\ell+1, \cdots x_M) + \psi(x_1, \cdots x_\ell - 1, \cdots x_M) = \sum_P A_P \left(\sum_{\ell=1}^{M} 2\cos k_{P\ell} \right) e^{i \sum_{\ell'=1}^{M} k_{P\ell'} x_{\ell'}}$$

$$= \left(2 \sum_{\ell=1}^{M} \cos k_\ell \right) \psi(x_1, \cdots x_M)$$

Taking into accout the $(S^3 S^3 - 1/4)$ part, we have

$$\boxed{E = J \sum_{j=1}^{M} \left(\cos k_\ell - \Delta \right)}$$

Magons only interact when they are neighors. The interaction will shift the momenta from the values of $\frac{2n\pi}{N}$ to other values, which is called phase shift. We need to determin the values of k_i.

① $M = 0$ $\longrightarrow$ vaccum, or, the reference state $|\uparrow\uparrow\cdots\uparrow\rangle$

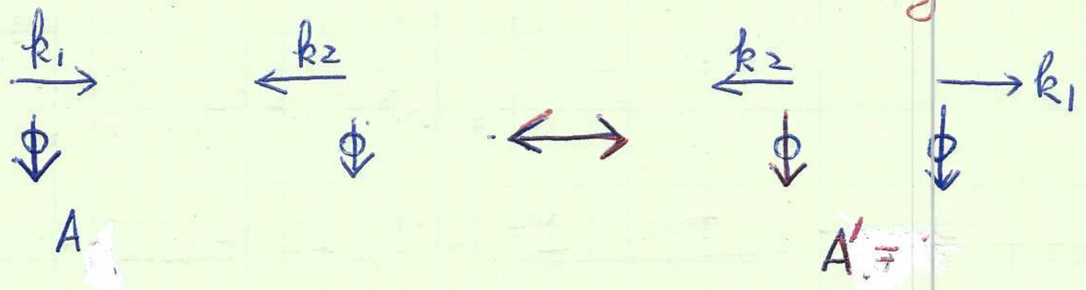
② $M=1$, free magnon, no interaction $k = \frac{2n\pi}{N}$.

If we consider the FM. case, this gives rise to the spin-wave state

$$E = |J|(1-\cos k) \approx \frac{|J|}{2}k^2.$$

for $\Delta=1$, the magon is gapless due to $SU(2)$ symmetry. The state with $k=0$, belong to the sector of $S=\frac{N}{2}$, $S_z=\frac{N}{2}-1$. The other $N-1$ states with $k\neq 0$, correspond to $S=S_z=\frac{N}{2}-1$. We say magons carry spin-1.

③ Now Let's check the 2-magon case. — interaction induces scattering



P=(1,2)

$$\boxed{\psi(x_1,x_2) = A e^{i(k_1x_1+k_2x_2)} + A' e^{i(k_2x_1+k_1x_2)}}$$ ← for $x_1 < x_2$

If x_1 and x_2 are not adjacent, the above $\psi(x_1,x_2)$ satisfies Eq (*),

i.e $\frac{J}{2}\left(\psi(x_1+1,x_2)+\psi(x_1-1,x_2)+\psi(x_1,x_2-1)+\psi(x_1,x_2+1)\right)-2J\Delta\psi(x_1x_2)$

$= E\psi(x_1x_2)$ (1).

Now consider the case that $x_1+1=x_2$, then we have

$$\frac{J}{2}\left(\psi(x_1-1,x_2)+\psi(x_1,x_2+1)\right)-J\Delta\psi(x_1,x_2)=E\psi(x_1,x_2) \quad (2)$$

compare (1) and (2), literally, we can take (1) by setting $x_2=x_1+1$,

then require

$$\frac{J}{2}\left(\psi(x_1+1,x_2)+\psi(x_1,x_2-1)\right)-J\Delta\psi(x_1x_2)=0. \quad (3)$$

Then we can use Eq(1) ~~to~~ to describe the WF at all cases, by requiring

$$\boxed{\frac{J}{2}\left(\psi(x_2,x_2)+\psi(x_1,x_1)\right)=J\Delta\psi(x_1,x_2)}$$

← by setting $x_2=x_1+1$ in Eq 3.

← boundary condition

$\Rightarrow \frac{1}{2}\left[Ae^{i(k_1+k_2)x_1}+A'e^{i(k_1+k_2)x_1}\right.$

$\left.+Ae^{i(k_1+k_2)x_2}+A'e^{i(k_1+k_2)x_2}\right]=\Delta\left[Ae^{i(k_1x_1+k_2x_2)}+A'e^{i(k_2x_1+k_1x_2)}\right]$

$\Rightarrow A\left(e^{i(k_1+k_2)}+1\right)+A'\left(e^{i(k_1+k_2)}+1\right)=2\Delta\left(Ae^{ik_2}+A'e^{ik_1}\right)$

$$\frac{A'}{A}=-\frac{e^{i(k_1+k_2)}-2\Delta e^{ik_2}+1}{e^{i(k_1+k_2)}-2\Delta e^{ik_1}+1},$$

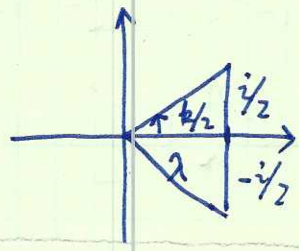
$$=-e^{i\Theta(k_2,k_1)}$$

A: k_1, k_2

A': $k_1'=k_2,\ k_2'=k_1$

★ For the isotropic case, i.e. $\Delta = 1$, we can parameterize

$$e^{ik_i} = \frac{\lambda_i + i/2}{\lambda_i - i/2} \Rightarrow \lambda_i = \frac{1}{2}\cot\frac{k_i}{2}$$



$$\frac{i/2}{\lambda} = \tan\frac{k}{2}$$

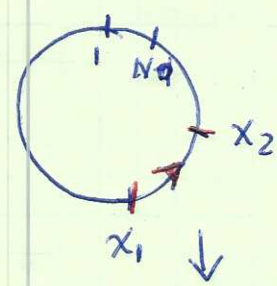
$$\Rightarrow \lambda = \frac{i}{2}\cot k/2$$

$$\boxed{\begin{aligned} e^{i\Theta(k_1,k_2)} &= \frac{1+e^{i(k_1+k_2)}-2e^{ik_1}}{1+e^{i(k_1+k_2)}-2e^{ik_2}} \\ &= -\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i} \end{aligned}}$$

Proof: $$e^{i\Theta(k_1,k_2)} = \frac{1+\dfrac{(\lambda_1+i/2)(\lambda_2+i/2)-2(\lambda_1+i/2)(\lambda_2-i/2)}{(\lambda_1-i/2)(\lambda_2-i/2)}}{1+\dfrac{(\lambda_1+i/2)(\lambda_2+i/2)-2(\lambda_1\mp i/2)(\lambda_2+i/2)}{(\lambda_1-i/2)(\lambda_2-i/2)}}$$

$$= \frac{(\lambda_1-i/2)(\lambda_2-i/2)+(\lambda_1+i/2)(\lambda_2+i/2)-2(\lambda_1+i/2)(\lambda_2-i/2)}{(\lambda_1-i/2)(\lambda_2-i/2)+(\lambda_1+i/2)(\lambda_2+i/2)-2(\lambda_1-i/2)(\lambda_2+i/2)}$$

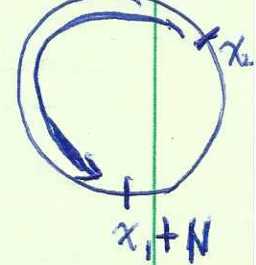
$$= \frac{i(\lambda_1-\lambda_2)-1}{i(\lambda_2-\lambda_1)-1} = -\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}.$$



★ peroidical boundary condition:

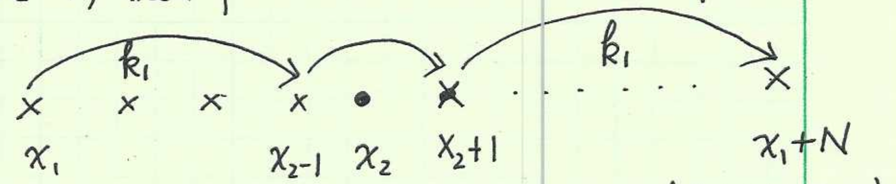
Set $x_1 \to x_1 + N$, and we want

$$\psi(x_1, x_2) = \psi(x_2, x_1+N) \leftarrow \text{since } x_1+N > x_2$$



- Let's elaborate what happens when we move particle 1 from $x_1 \to x_1+N$, during which particle 2 is fixed at x_2.

Trace the A-term, in which particle 1 carries momentum k_1. Let's move particle 1 from $x_1 \to x_2-1$, then particle 1 accumulates phase $e^{ik_1(x_2-x_1-1)}$.



- Then particle 1 hops from x_2-1, to x_2+1. Due to boson symmetry

$$e^{ik_1(x_2-x_1-1)} \cdot \frac{A'}{A} e^{ik_1 \cdot 2} \cdot e^{ik_1(x_1+N-x_2-1)}$$

$$= \frac{A'}{A} e^{ik_1 N}$$

$$\psi(x_2+1, x_2) = \psi(x_2, x_2+1)$$

but in order to trace the k_1-momentum, we need to continue to A'-term.

Hence $\quad A' e^{i(k_2 x_2 + k_1(x_2+1))} \quad \to$ we gain the phase $\frac{A'}{A} e^{ik_1 \cdot 2}$

- The next step, we move particle 1, to x_1+N, ~~Then the~~ the phase gained $e^{ik_1(x_1+N-x_2-1)}$.

Add all the phases ~~together~~ $\Rightarrow$ $\boxed{\frac{A'}{A} e^{ik_1 N} = 1}$ (*)

i.e. $\quad A e^{i(k_1 x_1 + k_2 x_2)} = A' e^{i(k_2 x_2 + k_1(x_1+N))}$

In other words, we need to count the phase shifts during the collisions as moving particle 1 from $x_1 \to x_2 \to x_1+N$.

We can also trace the A' term, in which particle 1 carries momentum $k_1' = k_2$ and particle 2 carries momentum $k_2' = k_1$.

We can repeat the above analysis $\Rightarrow$ $e^{ik_1'N} \dfrac{A}{A'} = 1$

or $\boxed{e^{ik_2N} \dfrac{A}{A'} = 1 \quad (**)}$ $\longrightarrow$ $A' e^{i(k_2x_1 + k_1x_2)} = A e^{i(k_1x_2 + k_2(x_2+N))}$.

Once Eq (*, **) are satisfied, its also satisfied $\psi(x_1, x_2) = \psi(x_2+N, x_1)$.

But if we want $x_2 \to x_2 + N$, then we need to be careful, since during this process, it collides with $x_1 + N$, we should set

$\boxed{\psi(x_2, x_1+N) = \psi(x_1+N, x_2+N)}$ ← which yields the same result as $\psi(x_1, x_2) = \psi(x_2, x_1+N)$, due to $e^{i(k_1+k_2)N} = 1$.

Global Shift:

$$\psi(x_1, x_2) \equiv \psi(x_1+N, x_2+N)$$

$$\Rightarrow e^{i(k_1+k_2)N} = 1$$

For 2-magon states, we have

$$e^{ik_1N} = \frac{A}{A'} = (-) e^{-i\Theta(k_2, k_1)} = (-) e^{i\Theta(k_1, k_2)}$$

$$e^{ik_2N} = \frac{A'}{A} = (-) e^{-i\Theta(k_1, k_2)} = (-) e^{i\Theta(k_2, k_1)}$$

For the isotropic case of $\Delta = 1$, plug in $e^{ik_i} = \dfrac{\lambda_i + i/2}{\lambda_i - i/2}$

$$\Rightarrow \boxed{\begin{aligned} \left(\frac{\lambda_1 + i/2}{\lambda_1 - i/2}\right)^N &= \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \\ \left(\frac{\lambda_2 + i/2}{\lambda_2 - i/2}\right)^N &= \frac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i} \end{aligned}}$$

- Prove the existence of bound states for $M = 2$.

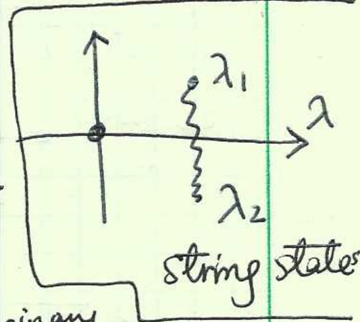
Bound states correspond to complex solutions of k, i.e. as $N \to \infty$

$\left(\dfrac{\lambda_1 + i/2}{\lambda_1 - i/2}\right)^N \to 0$, or, ∞, which means $\lambda_1 - \lambda_2 = \pm i$.

$$E = J[\cos k_1 + \cos k_2 - 2] = \frac{J}{2}\left[e^{ik_1} + e^{-ik_1} + e^{ik_2} + e^{-ik_2} - 4\right]$$

$$= \frac{J}{2}\left[\frac{\lambda_1 + i/2}{\lambda_1 - i/2} + \frac{\lambda_1 - i/2}{\lambda_1 + i/2} + \frac{\lambda_2 + i/2}{\lambda_2 - i/2} + \frac{\lambda_2 - i/2}{\lambda_2 + i/2} - 4\right] = J\left[\frac{\lambda_1^2 - 1/4}{\lambda_1^2 + 1/4} + \frac{\lambda_2^2 - 1/4}{\lambda_2^2 + 1/4} - 2\right]$$

$$= -J/2\left[\frac{1}{\lambda_1^2 + 1/4} + \frac{1}{\lambda_2^2 + 1/4}\right]$$



set $\left.\begin{aligned} \lambda_1 &= x + \tfrac{i}{2} \\ \lambda_2 &= x - i/2 \end{aligned}\right\} \Rightarrow E = -\dfrac{J}{2}\left[\dfrac{1}{x^2 + ix} + \dfrac{1}{x^2 - ix}\right] = -\dfrac{J}{2}\,\dfrac{1}{x^2 + 1}$

if E is real, we need either X real number, or purely imaginary.

If X is purely imaginary, then both λ_1, λ_2 are imaginary $\Rightarrow$ k_1, k_2 purely imaginary, which can be tested not the case. Hence X is real, and $\lambda_1 = \lambda_2^*$. In this case $e^{i(k_1 + k_2)} = \dfrac{x + i}{x - i}$ $\Rightarrow$ $\boxed{E = \dfrac{J}{2}\left[\cos(k_1 + k_2) - 1\right]}$

$$\cos(k_1 + k_2) = \frac{x^2 - 1}{x^2 + 1}$$

hence the bound state energy is determined by the center of mass momentum.

- For scattering states, k_1, k_2 are real, it can be proved that

$$\frac{1-\cos(k_1+k_2)}{2} \leqslant (1-\cos k_1) + (1-\cos k_2)$$

- $\Rightarrow$ For FM case, $J<0$,

$$E_{bound} = |J| \left[1 - \cos(k_1+k_2) \right]$$

$$E_{scattering} = |J| \left[(1-\cos k_1) + (1-\cos k_2) \right] = |J| \left[2 - 2\cos\frac{k_1+k_2}{2} \cos\frac{k_1-k_2}{2} \right]$$

$\Rightarrow E_{bound} < E_{scattering}$

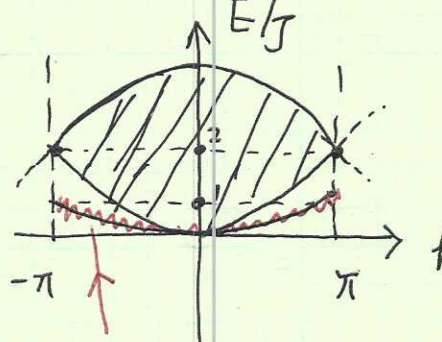
For the AFM case $J>0 \Rightarrow E_{bound\ state} > E_{scattering}$.

- For the FM case, the upper / lower boundary of the $E_{scattering}$ state

$$E_{\pm}(k) = 2|J| \left(1 \pm \cos\frac{k}{2}\right)$$

$$E_{bound}(k) = \frac{|J|}{2}(1-\cos k)$$

where k is the center of mass momentum



2-string states or magnon bound states.

- Many-body version in the AFM system.

W. Yang, et al arxiv 1702.01854

Z. Wang et al, Nature 554, 219 (2018)

$\boxed{\text{• HW: perform numerical solutions to all the 2-magnon states.}}$