Let me first present a general process using real space OPE to derive RG.

$S[\phi] = S_0[\phi] + \sum_i g_i \int d^dx \, O_i(x)$, where d^dx means $d\tau\, dx$ here

operator $O_i(x)$ has an naive scaling dimension d, but its correlation gives

$$\langle O(x) O(y) \rangle \sim \frac{a_0^{-2d}}{|(x-y)/a_0|^{2\Delta}},$$ where a_0 is the short range cut off.

Let us look at $\int d^dx \, O_i(x) = \sum_x a_0^d \, O_i(x)$ (change to discretized Rep)

Now let us change the cut off to $a = \lambda a_0$ $(\lambda > 1)$ and

$$\langle a^d O'(x)\, a^d O'(x) \rangle \sim \frac{1}{|x-y/a|^{2\Delta}},$$ where $O'(x)$ the coarse averaged version of $O(x)$

Compare with

$$\langle a_0^d O(x)\, a_0^d O'(x) \rangle \sim \frac{1}{|(x-y)/a_0|^{2\Delta}}$$

$$\Rightarrow \quad a^d O'(x) / a_0^d O(x) = \left(\frac{a}{a_0}\right)^{\Delta}, \quad \text{or} \quad O'(x) \simeq \lambda^{\Delta-d} O(x),$$

up to first order

$$g \int d^dx \, O(x) = g \sum_x a_0^d O(x) \xrightarrow{\text{average}} g \lambda^{\Delta-d} \sum_x a^d O'(x)$$

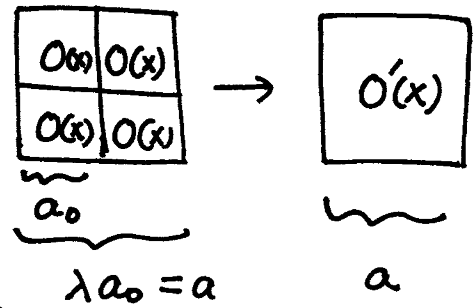
$$= g \lambda^{d-\Delta} \int d^dx \, O'(x)$$

the miscro-sopic length scale, or, the lattice const $a_0 \to a = \lambda a_0$

thus at the tree level $g(\lambda) = g\,\lambda^{d-\Delta} \Rightarrow dg(\lambda) = (d-\Delta)\,\lambda^{d-\Delta-1}$

$$\boxed{\frac{dg}{d\ln\lambda} = (d-\Delta)\,g}.$$

how to understand this?



$$O'(x) = \ = \langle O(x)\rangle_{\lambda a_0} = \frac{1}{(\lambda a_0)^d} \cdot O(x)(\lambda a_0)^{\Delta}$$

表观体积/面积 scaling　　真实 dimension scaling

$$\sim O(x)\,\lambda^{\Delta-d}.$$

now go to one loop, $e^{-S_{eff}(\phi)} = \int D\phi_> e^{-S_0(\phi_<)}\, e^{-S_0(\phi_>)}\, e^{-S_{int}(\phi_>,\phi_<)}$

$$= e^{-S_0(\phi_<)} \langle e^{-S_{int}(\phi_>,\phi_<)}\rangle_{fast}$$

$$\langle e^{-S_{int}(\phi_>,\phi_<)}\rangle_{fast} = e^{-\left[\langle S_{int}(\phi_>,\phi_<)\rangle_{fast} - \frac{1}{2!}\left(\langle S_{int}^2\rangle_{fast} - \langle S_{int}\rangle_{fast}^2\right) + \cdots\right]}$$

we use $\langle\ \rangle_{fast}$ for averaging over fast mode.

Fast or slow modes refer to $\phi = \underset{fast}{\phi_>} + \underset{slow}{\phi_<}$.

we have already showed $\langle S_{int}\rangle_{fast} \Rightarrow g_i \int d^dx \langle O_i(\phi_>,\phi_<)\rangle_{fast} = g_i \lambda^{d-\Delta} \int d^dx\, O_i(\phi_<)$.

~~Suppose we have OPE~~

$$O_i(x)\, O_j(y) \longrightarrow \sum_k C_{ij}^k\, O_k(y) \frac{a_0^d}{\left|\frac{x-y}{a_0}\right|^{\Delta_i+\Delta_j-\Delta_k}}$$

valid for $|x-y| > a_0$.

$$\langle O_i(x)\, O_j(y)\rangle_{fast} = \sum_k C_{ij}^k \langle O_k(y)\rangle_{fast} \frac{a_0^d}{\left|\frac{x-y}{a_0}\right|^{\Delta_i+\Delta_j-\Delta_k}}\, \Theta(a_0 < |x-y| < \lambda a_0)$$

if $|x-y| > \lambda a_0$, then $O_i(x)\, O_j(y)$ are related by the slow mode, which are not taken into account by $\langle\ \rangle_{fast}$.

$$S_{eff}(\phi) = S_0(\phi_<) + g_i \sum_i \int d^dx \langle O(\phi)\rangle_> - \frac{1}{2}\sum_{ij} g_i g_j \int d^dx\, d^dy \langle O_i(x)\, O_j(y)\rangle_>$$

$$\sum_{ij} g_i g_j \int d^dx\, d^dy \langle O_i(x)\, O_j(y)\rangle_> = \sum_k C_{ij}^k\, g_i g_j \int d^dx\, d^dy \langle O_k(y)\rangle_> \frac{a_0^{-d}}{\left|\frac{(x-y)}{a_0}\right|^{\Delta_i+\Delta_j-\Delta_k}}$$

$\int d^dx \to \int d^d(x-y)$ integrate over $a_0 \to \lambda a_0$

$$\int_{a_0}^{\lambda a_0} d^d(x-y) \frac{a_0^{-d}}{|(x-y)/a_0|^{\Delta_i+\Delta_j-\Delta_k}} = S_{d-1}\int_1^{\lambda} r^{d-1}\, r^{\Delta_i+\Delta_j-\Delta_k}\, dr$$

$$= S_{d-1} \frac{\lambda^{d-(\Delta_i+\Delta_j-\Delta_k)} - 1}{d-(\Delta_i+\Delta_j-\Delta_k)} \xrightarrow[\lambda\to 1]{} S_{d-1}\ln\lambda$$

or $\{\ \ln\lambda$ if $d = \Delta_i+\Delta_j-\Delta_k$

S_{d-1} is the area of sphere of $d-1$ dimensional.

$$\Rightarrow \sum_{ij} \hat{g}_i g_j \int d^dx \int d^dy \langle O_i(x) O_j(y) \rangle_> = \sum_K \sum_{ij} C_{ij}^k g_i g_j \int d^dy \langle O_K(y) \rangle_> S_{d-1} \ln\lambda$$

$$= \sum_K \sum_{ij} C_{ij}^K g_i g_j \underline{\lambda^{d-\Delta_K}} S_d \ln\lambda \int d^dy \; O_K(\phi_<(y))$$

$$\xrightarrow[\lambda\to 1]{} 1 + (d-\Delta_K)\ln\lambda$$

thus its effect can be droped

$$\Rightarrow S_{eff}(\phi) = S_0(\phi_<) + \sum_K \left[g_K \lambda^{d-\Delta} - \frac{1}{2} \sum_{ij} C_{ij}^K g_i g_j S_{d-1} \ln\lambda \right] \int d^dy \, O_K(\phi_<)$$

$$\Rightarrow \quad g_K' = g_K \lambda^{d-\Delta} - \frac{1}{2} \sum_{ij} C_{ij}^K g_i g_j S_{d-1} \ln\lambda$$

$$= g_K + \left[(d-\Delta) - \frac{1}{2} \sum_{ij} C_{ij}^K g_i g_j S_{d-1} \right] \ln\lambda$$

$$\boxed{\Rightarrow \frac{d g_K}{d \ln\lambda} = (d-\Delta) - \frac{1}{2} \sum_{ij} C_{ij}^K g_i g_j S_{d-1}}$$

RG for $H = \int dx \, \frac{v}{2}\left[K\Pi^2 + \frac{1}{K}(\partial_x\phi)^2\right] + \frac{g}{2(\pi a)^2}\cos\beta\phi(x)$

$$\to \mathcal{L} = \int dx \, \frac{1}{2K}\left[\frac{1}{v}(\partial_t\phi)^2 - v(\partial_x\phi)^2\right] - \frac{g}{2(\pi a)^2}\cos\beta\phi(x)$$

$$\varphi(x) = \sqrt{\frac{1}{4\pi}}\sum_{q\neq 0}\sqrt{K/n_q}\,\left(\tilde{b}_q e^{iqx} + \tilde{b}^+_q e^{-iqx}\right) + \frac{\sqrt{\pi}}{L}\sqrt{K}\left(x(\tilde{N}_R+\tilde{N}_L) - vt(\tilde{N}_R-\tilde{N}_L)\right)$$

$$\Theta(x) = \sqrt{\frac{1}{4\pi}}\sum_{q\neq 0}\frac{1}{\sqrt{Kn_q}}\,\mathrm{sgn}(q)\left(\tilde{b}_q e^{iqx} + \tilde{b}^+_q e^{-iqx}\right) + \frac{\sqrt{\pi/K}}{L}\left[x(\tilde{N}_R-\tilde{N}_L) - vt(\tilde{N}_R+\tilde{N}_L)\right]$$

which satisfies $[\varphi(x), -\partial_x\Theta(x')] = \delta(x-x')$

or $\Pi_x = -\partial_x\Theta$

useful identity

$$\boxed{\begin{aligned}\langle\varphi(x)\varphi(x') - \varphi^2(0)\rangle &= \frac{-K}{4\pi}\left[\ln\frac{a-i(\Delta x - v\Delta t)}{a} + \ln\frac{a+i(\Delta x+v\Delta t)}{a}\right] \\ &= -\frac{K}{4\pi}\ln\frac{(a+iv\Delta t)^2+(\Delta x)^2}{a^2} \\ \langle\Theta(x)\Theta(x') - \Theta^2(0)\rangle &= -\frac{1}{4\pi K}\ln\frac{(a+iv\Delta t)^2+(\Delta x)^2}{a^2}\end{aligned}}$$

$\tau = it$ imagniary time

$$z = -i(x-vt) = v\tau - ix \qquad \bar{z} = i(x+vt) = v\tau + ix$$

$$\partial_z = -\frac{i}{2}\left[\frac{1}{v}\partial_t - \partial_x\right] = \frac{1}{2}\left[\frac{1}{v}\partial_\tau + i\partial_x\right], \quad \partial_{\bar{z}} = -\frac{i}{2}\left[\frac{1}{v}\partial_t + \partial_x\right] = \frac{1}{2}\left[\frac{1}{v}\partial_\tau - i\partial_x\right]$$

$$\partial_x = -i(\partial_z - \partial_{\bar{z}}), \quad \partial_t = iv(\partial_z + \partial_{\bar{z}})$$

$$\partial_t^2 - \partial_x^2 = -(\partial_z + \partial_{\bar z})^2 + (\partial_z - \partial_{\bar z})^2 = -4\,\partial_z \partial_{\bar z}$$

$$\mathcal{L}_{free} = \frac{1}{2K}\left[(\partial_t \phi)^2 - (\partial_x \phi)^2\right] = \frac{1}{2K}\left(-4\,\partial_z \phi\, \partial_{\bar z}\phi\right).$$

Let us consider the OPE $\cos\beta\phi(x)\,\cos\beta\phi(x')$

$$\to \left(e^{i\beta\phi(x)} + e^{-i\beta\phi(x)}\right)\left(e^{i\beta\phi(x')} + e^{-i\beta\phi(x')}\right)$$

$$e^{i\beta\phi(x)}\, e^{i\beta\phi(x')} + e^{-i\beta\phi(x)}\, e^{-i\beta\phi(x')} \underset{x\to x'}{=} e^{2i\beta\phi(x)} + e^{-2i\beta\phi(x)}$$

$$= 2\cos 2\beta\phi(x)$$

because $[\phi(x), \phi(x')] = 0$.

$$e^{i\beta\phi(x)}\, e^{-i\beta\phi(x')} = :e^{i\beta(\phi(x) - \phi(x'))}:\; e^{\beta^2\langle \phi(x)\phi(x') - \phi^2(0)\rangle}$$

using $e^A e^B = :e^{A+B}:\; e^{\langle G|\frac{A^2}{2} + \frac{B^2}{2} + AB|G\rangle}$

$$\phi(x) - \phi(x') = \partial_z\phi\, \varepsilon + \partial_{\bar z}\phi\, \bar\varepsilon + \frac{1}{2!}\left(\partial_z^2\varphi\, \varepsilon^2 + \partial_{\bar z}^2\varphi\, \bar\varepsilon^2 + 2\partial_z\varphi\,\partial_{\bar z}\varphi\, \varepsilon\bar\varepsilon\right)$$

$$e^{\beta^2\langle \phi(x)\phi(x') - \phi^2(0)\rangle} = e^{-\frac{\beta^2 K}{4\pi}\ln\frac{\varepsilon\bar\varepsilon}{a^2}}.$$

$$:e^{i\beta(\phi(x) - \phi(x'))}: + :e^{-i\beta(\phi(x) - \phi(x'))}:$$

$$= 1 + i\beta[\partial_z\phi\, \varepsilon + \partial_{\bar z}\phi\, \bar\varepsilon] - \frac{\beta^2}{2}[\partial_z\phi\, \varepsilon + \partial_{\bar z}\phi\, \bar\varepsilon]^2 + \frac{i\beta}{2}[\partial_z^2\varphi\, \varepsilon^2 + \partial_{\bar z}^2\varphi\, \bar\varepsilon^2 + 2\partial_z\varphi\,\partial_{\bar z}\varphi\, \varepsilon\bar\varepsilon]$$

$$+ 1 - i\beta[\partial_z\phi\, \varepsilon + \partial_{\bar z}\phi\, \bar\varepsilon] - \frac{\beta^2}{2}[\partial_z\phi\, \varepsilon + \partial_{\bar z}\phi\, \bar\varepsilon]^2 - \frac{i\beta}{2}[\partial_z^2\varphi\, \varepsilon^2 + \dots]$$

$$= 2 - \beta^2\left[(\partial_z\phi)^2\varepsilon^2 + (\partial_{\bar z}\phi)\bar\varepsilon^2 + 2(\partial_z\phi)(\partial_{\bar z}\phi)\,\varepsilon\bar\varepsilon\right]$$

$$\Rightarrow e^{i\beta\phi(x)}\, e^{-i\beta\phi(x')} + e^{-i\beta\phi(x)}\, e^{i\beta\phi(x')}$$

$$= \left(2 - \beta^2\Big[\underbrace{(\partial_z\phi)^2 \varepsilon^2 + (\partial_{\bar z}\phi)^2 \bar\varepsilon^2}_{\text{not in original Hamiltonian, drop away}} + 2(\partial_z\phi)(\partial_{\bar z}\phi)\varepsilon\bar\varepsilon\Big)\right] \cdot \left(\frac{a^2}{\varepsilon\bar\varepsilon}\right)^{\frac{\beta^2 k}{4\pi}}$$

$$\rightarrow \quad -2\beta^2 \underbrace{(\partial_z\phi\,\partial_{\bar z}\phi)}\, (\varepsilon\bar\varepsilon)^{1-\frac{\beta^2 k}{4\pi}}\, (a^2)^{\frac{\beta^2 k}{4\pi}}$$

$$\Rightarrow \quad S(\phi) = \int d^2x \ \frac{1}{2k}\left[4\,\partial_z\phi\,\partial_{\bar z}\phi\right] + \frac{g}{2(\pi a)^2}\cos\beta\phi$$

the second order :

$$-\frac{1}{2}\,\frac{g^2}{4(\pi a)^4\cdot 4}\int d^2x\, d^2y \ (-2\beta)^2 \left(\partial_z\phi\,\partial_{\bar z}\phi\right)(a^2)^{\frac{\beta^2 k}{4\pi}}\,(\varepsilon\bar\varepsilon)^{1-\frac{\beta^2 k}{4\pi}}$$

$$= \frac{g^2}{(2\pi a)^4}\int d^2x\ \beta^2 (a^2)^{\frac{\beta^2 k}{4\pi}}\left(\partial_z\phi\,\partial_{\bar z}\phi\right)\int d^2(x-y)\ (\varepsilon\bar\varepsilon)^{1-\frac{\beta^2 k}{4\pi}}$$

$$S(1)\int_a^{\lambda a} r^2\, dr\ \ r^{2\left(1-\frac{\beta^2 k}{4\pi}\right)}$$

$$= 2\pi \ln\lambda \ a_0^{4-\frac{\beta^2 k}{4\pi}}$$

$$\downarrow$$

$$\frac{\ln\lambda\,\beta^2 g^2}{(2\pi)^3}\int d^2x \ \ \partial_z\varphi\,\partial_{\bar z}\varphi$$

$$\Rightarrow \text{a correction to} \quad \frac{2}{K} \rightarrow \frac{2}{K} + \frac{\beta^2 g^2}{(2\pi)^3}\ln\lambda$$

$$\Rightarrow \quad \frac{d\,\frac{2}{K}}{d\ln\lambda} = \frac{\beta^2 g^2}{(2\pi)^3}\ , \qquad d$$

the RG for "g" appears at the tree level, we can directly read out

$$\frac{dg}{d\ln\lambda} = (d-\Delta)g = \left(d-\frac{\beta^2 K}{4\pi}\right)g$$

$$\Rightarrow \boxed{\begin{aligned} \frac{d\frac{2}{K}}{d\ln\lambda} &= \frac{\beta^2 g^2}{(2\pi)^3} \\ \frac{dg}{d\ln\lambda} &= \left(d-\frac{\beta^2 K}{4\pi}\right)g \end{aligned}}$$

for the spinless fermion case

$\beta = \sqrt{16\pi}$

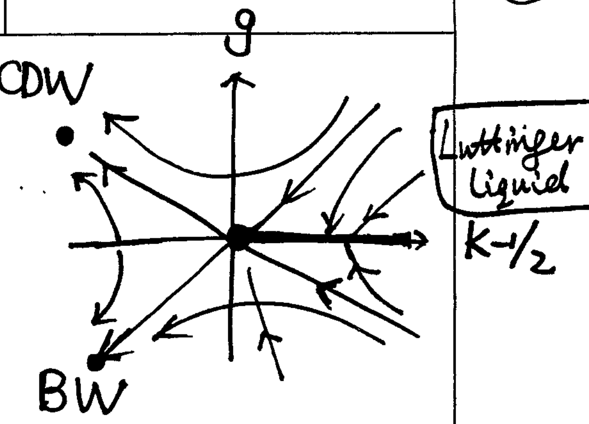
$$\begin{cases} \dfrac{dg}{d\ln\lambda} = 4\left(\frac{1}{2}-K\right)g \\ \dfrac{d\left(\frac{1}{2}-K\right)}{d\ln\lambda} \underset{K\to 1/2}{\simeq} \dfrac{K^2 g^2}{\pi^2} \approx \dfrac{g^2}{4\pi^2} \end{cases}$$

$\Rightarrow$

$$\frac{g\,dg}{\left(\frac{1}{2}-K\right)d\left(\frac{1}{2}-K\right)} = \frac{4\pi^2}{K^2} = 16\pi^2$$

or $\quad g^2 - 16\pi^2\left(\frac{1}{2}-K\right)^2 = \text{const}$

RG flow:



if g is relavant $\Rightarrow$ $\cos\sqrt{16\pi}\phi$ is pinned

which can be achived by choosing

if $g>0$, $V>0$ $\Rightarrow$

$$\cos\sqrt{16\pi}\phi = -1 \Rightarrow \sqrt{16\pi}\phi = \pm\pi, \pm 3\pi, \ldots$$

$$\sqrt{4\pi}\phi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

$\Rightarrow$ $\mathrm{Re}\, N^{+} = \psi_L^{+}\psi_R + \psi_R^{+}\psi_L = \frac{1}{\pi a}\sin\sqrt{4\pi}\phi$ is pinned $\Rightarrow$ on site CDW

$\mathrm{Im}\, N^{+} = i(\psi_L^{+}\psi_R - \psi_R^{+}\psi_L) \sim \frac{1}{\pi a}\cos\sqrt{4\pi}\phi$ is not pinned No BW order.

This agree with intuition!

if the bare value of $g<0$, $g\to-\infty$, then $\cos\sqrt{16\pi}\phi = 1$

$$\Rightarrow \sqrt{16\pi}\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$$

$\sqrt{4\pi}\phi = 0, \pm\pi, \pm 2\pi \ldots \Rightarrow$ $\mathrm{Im}\, N^{+}$ is pinned $\Rightarrow$ BW (bond ordering)

if we choose $V<0$, its fine for g, but it will also push K to large value. If K can be made $K<\frac{1}{2}$ by other longer range interaction, we could have BW phase.

~~RG for Sine-Gordon~~

$$\mathcal{Z} = \int_\Lambda D\phi \; e^{-S} \quad \text{where } S = S_0 + S_1 = \frac{1}{2}\int d^2x \, (\nabla\phi)^2 + \frac{g}{(2\pi a)^2}\cos\beta\phi$$

① divide $\phi(x)$ into fast mode $h(x)$ and slow varying mode $\phi_{\Lambda'}(x)$

Λ' is a smaller cutoff than Λ

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_{k<\Lambda'} e^{ikx}\phi_k + \frac{1}{\sqrt{V}}\sum_{\Lambda'<k<\Lambda} e^{ikx}\phi_k = \phi_{\Lambda'}(x) + h(x)$$

$$\Rightarrow \mathcal{Z} = \int D\phi_{\Lambda'} \, Dh \quad e^{-S_0(\phi_{\Lambda'})} \frac{e^{-S_0(h)} \; e^{-S_1[\phi_{\Lambda'}+h]}}{\int Dh \; e^{-S_0(h)}}$$

← just a const

$$= \int D\phi_{\Lambda'} \; e^{-S_0(\phi_{\Lambda'})} \left\langle e^{-S_1(\phi_{\Lambda'}+h)} \right\rangle_h$$

(average over fast mode)

$$\left\langle e^{-S_1(\phi_{\Lambda'}+h)} \right\rangle_h \simeq e^{-\{\langle S_1\rangle_h + \frac{1}{2}[-\langle S_1^2\rangle_h + \langle S_1\rangle_h^2]\}}$$

← Cumulant expansion

At first order

$$\left\langle e^{\pm i\beta h(x)} \right\rangle_h = e^{-\frac{\beta^2}{2}\langle h^2(x)\rangle_h} = \exp\left[-\frac{\beta^2}{2}\int \frac{d^2k}{(2\pi)^2}\langle h(k)\,h(-k)\rangle_h\right]$$

$$= \exp\left[-\frac{\beta^2}{2}\int_{\Lambda'<k<\Lambda} \frac{k\,dk}{2\pi} \cdot \frac{1}{k^2}\right] = e^{+\frac{\beta^2}{4\pi}\ln\frac{\Lambda'}{\Lambda}} \approx 1 - \frac{\beta^2}{4\pi}\,dl$$

where $\Lambda' = \Lambda e^{-dl} \Rightarrow \ln\left(\frac{\Lambda'}{\Lambda}\right) = \ln[e^{-dl}] = -dl$

$$\Rightarrow \langle S_1(\phi_{\Lambda'}+h)\rangle_h = \frac{g}{(2\pi a)^2}\int d^2x \, \langle \cos\beta(\phi_{\Lambda'}+h)\rangle_h$$

$$= \frac{g}{2(2\pi a)^2}\int d^2x \, \langle e^{i\beta(\phi_{\Lambda'}+h)} + e^{-i\beta(\phi_{\Lambda'}+h)}\rangle_h$$

$$= \frac{g}{2(2\pi a)^2}\int d^2x \, \left(1-\frac{\beta^2}{4\pi}dl\right)\cdot 2\cos\beta\phi_{\Lambda'} = \frac{g\left(1-\frac{\beta^2}{4\pi}dl\right)}{(2\pi a)^2}\int_{\Lambda'} d^2x \, \cos\beta\phi_{\Lambda'}$$

(2) At the second order

$$\langle S_1^2(\phi_{\Lambda'}+h)\rangle_h$$

$$= \frac{g^2}{(2\pi a)^4 2^2}\int d^2x_1 d^2x_2 \, \Big[\Big\langle e^{i\beta[(\phi_{\Lambda'}+h)(x_1)+(\phi_{\Lambda'}+h)(x_2)]} + e^{-i\beta[\phi_{\Lambda'}+h](x_1)-i\beta[\phi_{\Lambda'}+h](x_2)}$$
$$+ e^{i\beta[\phi_{\Lambda'}+h](x_1)-[\phi_{\Lambda'}+h](x_2)} + e^{-i\beta[\phi_{\Lambda'}+h](x_1)+i\beta[\phi_{\Lambda'}+h](x_2)} \Big\rangle_h$$

$$= \frac{g^2}{(2\pi a)^4\cdot 2^2}\int d^2x_1 d^2x_2 \quad e^{i\beta[\phi_{\Lambda'}(x_1)+\phi_{\Lambda'}(x_2)]} \langle e^{i\beta(h(x_1)+h(x_2))}\rangle_h$$
$$+ e^{-i\beta[\phi_{\Lambda'}(x_1)+\phi_{\Lambda'}(x_2)]} \langle e^{-i\beta(h(x_1)+h(x_2))}\rangle_h$$
$$+ e^{i\beta[\phi_{\Lambda'}(x_1)-\phi_{\Lambda'}(x_2)]} \langle e^{i\beta(h(x_1)-h(x_2))}\rangle_h$$
$$+ e^{-i\beta[\phi_{\Lambda'}(x_1)-\phi_{\Lambda'}(x_2)]} \langle e^{-i\beta(h(x_1)-h(x_2))}\rangle_h$$

$\langle S_1(\phi_{\Lambda'}+h)\rangle_h^2$ can be written in a similar form, but

replace $\langle e^{i\beta h(x_1)+i\beta h(x_2)}\rangle_h$ with $\langle e^{i\beta h(x_1)}\rangle_h \langle e^{i\beta h(x_2)}\rangle_h$

etc.

$$\langle e^{i\beta[h(x_1)+h(x_2)}\rangle_h - \langle e^{i\beta h(x_1)}\rangle_h \langle e^{i\beta h(x_2)}\rangle_h$$

$$= e^{-\frac{\beta^2}{2}[\langle h(x_1)+h(x_2)\rangle^2]} - e^{-\frac{\beta^2}{2}[\langle h^2(x_1)\rangle + \langle h^2(x_2)\rangle]}$$

$$= e^{-\beta^2\langle h^2(0)\rangle}\left[e^{-\beta^2\langle h(x_1)h(x_2)\rangle} - 1\right]$$

Similarly, collect all the terms $\Rightarrow$

$$\langle S_1^2(\phi_{\Lambda'}+h)\rangle_h - \langle S_1(\phi_{\Lambda'}+h)\rangle_h^2$$

$$= \frac{g^2}{2(2\pi a)^4}\int dx_1 dx_2 \; e^{-\beta^2\langle h^2(0)\rangle}\Big[\cos\beta(\phi_{\Lambda'}(x_1)+\phi_{\Lambda'}(x_2))\{e^{-\beta^2\langle h(x_1)h(x_2)\rangle}-1\}$$
$$+\cos\beta[\phi_{\Lambda'}(x_1)-\phi_{\Lambda'}(x_2)]\{e^{+\beta^2\langle h(x_1)h(x_2)\rangle}-1\}\Big]$$

the first term contribute to $\underbrace{\cos 2\beta\phi}$ and will be dropeel,

high rank, with large scaling dimensi

The second term

$$\cos\beta[\phi_{\Lambda'}(x_1)-\phi_{\Lambda'}(x_2)] \approx 1-\frac{1}{2}\beta^2\left[\vec{r}\cdot\nabla_R\,\phi_{\Lambda'}\right]^2, \text{ where } \vec{r}=\vec{X}_1-\vec{X}_2, \quad \vec{R}=\frac{\vec{X}_1+\vec{X}_2}{2}$$

$e^{\beta^2\langle h(x_1)h(x_2)\rangle_h}-1$ is a short-range function of r

from $e^{-\beta^2\langle h^2(0)\rangle_h}$

$$\rightarrow \frac{g^2}{2\cdot(2\pi a)^4}\int d\vec{R}\int d\vec{r}\ \left(1-\frac{\beta^2}{2\pi}dl\right)\cdot\left(-\frac{\beta^2}{2}\right)\left[\vec{r}\cdot\nabla_R\phi_\Lambda\right]^2\left(\beta^2\langle h(\vec{r})h(0)\rangle_h\right)$$

the intergal over $\vec{r}$ independent on direction of $\vec{r}$

$$\int d\vec{r}\ (\vec{r}\cdot\nabla_R\phi_\Lambda)^2\langle h(\vec{r})h(0)\rangle_h$$

$$= \int d\vec{r}\ \left[\left((\nabla_R^x\phi_\Lambda)^2(x^2)+(\nabla_R^y\phi_\Lambda)^2(y^2)\right.\right.$$
$$\left.+\,2\nabla_R^x\phi_\Lambda\nabla_R^y\phi_\Lambda\,xy\right]\langle h(\vec{r})h(0)\rangle_h$$

$$=\frac{1}{2}(\nabla_R\phi_\Lambda)^2\int d\vec{r}\ \ r^2\langle h(\vec{r})h(0)\rangle_h$$

$$\Rightarrow \frac{g^2}{2(2\pi a)^4}\int d\vec{R}\ \underbrace{\left(1-\frac{\beta^2}{2\pi}dl\right)}_{(\nabla_R\phi_\Lambda)^2}\left(-\frac{\beta^2}{2}\right)\left(\frac{\beta^2}{2}\right)\boxed{\int d\vec{r}\ \ r^2\langle h(\vec{r})h(0)\rangle_h}$$

calculated it!

$$\int d\vec{r}\ r^2\langle h(\vec{r})h(0)\rangle = \int_{\Lambda'<k<\Lambda}\frac{d^2k}{(2\pi)^2}\int d^2\vec{r}\ \frac{r^2}{k^2}e^{ikr\cos\theta-r0^+}$$

$$=\int d^2\vec{r}\ \ r^2\cdot\frac{1}{(2\pi)^2}\int_{\Lambda'<k<\Lambda}\frac{dk}{k}\int_0^{2\pi}d\theta\ \ e^{i\Lambda r\cos\theta-r0^+}$$

$$= \int d^2\vec{r}\ r^2 \int_0^{2\pi} d\theta\ e^{i\Lambda r\cos\theta - 0^+ r} \quad \frac{1}{(2\pi)^2} \ln\frac{\Lambda}{\Lambda'}$$

$$= \frac{dl}{2\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} r^3 dr\ e^{i\Lambda r\cos\theta - 0^+ r}$$

$$= \frac{dl}{2\pi\Lambda^4} \int_0^{2\pi} d\theta \frac{1}{[\cos\theta + i0^+]^4} \underbrace{\int_0^{+\infty} x^3 dx'\ e^{i(x' + i0^+)}} \qquad \Gamma(4) = 6$$

$$= \frac{3\,dl}{\pi\Lambda^4} \cdot 8\pi$$

need check

$$\int_0^{2\pi} d\theta \frac{1}{[\cos\theta + i0^+]^4} = 8\pi$$

$$= \frac{24\,dl}{\Lambda^4}$$

using $\Lambda = \frac{2\pi}{a}$ $\Rightarrow$ $\int dr\ \vec{r}^2 \langle h(r) h(0) \rangle_h = \frac{24 a^4}{16\pi^4} dl = \frac{3a^4}{2\pi^4} dl$

$\Rightarrow$ we get at the 2nd order.

$$\frac{1}{2} \frac{g^2}{2(2\pi a)^4} \left(1 - \frac{\beta^2}{2\pi} dl\right) \frac{\beta^4}{4} \cdot \frac{3a^4}{2\pi^4} dl \int d\vec{R}\ (\nabla_R \phi)^2$$

$$= dl \frac{3a^4}{32\pi^4} \frac{g^2\beta^4}{(2\pi a)^4} \int dR^2 (\nabla_R\phi)^2$$

$$= \frac{1}{2} A g^2 \beta^4 dl \int dR^2 (\nabla_R\phi)^2$$ where A is a dimensionless const depending on the cutoff.

$$\Rightarrow S_{eff} = \frac{g\left(1-\frac{\beta^2}{4\pi}dl\right)}{(2\pi a)^2}\int_{a'} d^2x \cos\beta\phi_{\Lambda'}$$

$$+ \frac{1}{2}\left(1+Ag^2\beta^4 dl\right)\int_{a'} d^2x \left(\nabla\phi_{\Lambda'}\right)^2$$

next step: restore the cut off $x' = (1-dl)\,x$, and thus the cut off for x', comes back to $a'(1-dl)=a$, or $\Lambda' \to \Lambda$.

$$d^2x = d^2x' \,(1+2dl) \qquad \nabla_x^2 = \nabla_{x'}^2\,(1-2dl)$$

$$\Rightarrow S_{eff} = \frac{g\left(1+\left(2-\frac{\beta^2}{4\pi}\right)dl\right]}{(2\pi a)^2}\int_a d^2x' \cos\beta\phi_{\Lambda'}$$

$$+\frac{1}{2}\left(1+Ag^2\beta^4 dl\right)\int_a d^2x \left(\nabla\phi_\Lambda\right)^2$$

$\to$ rescale the field $\phi'_\Lambda = \sqrt{1+Ag^2\beta^2 dl}\;\phi_\Lambda$

$$\to S_{eff} = \frac{1}{2}\int_a d^2x \left(\nabla\phi'_\Lambda\right)^2 + \frac{g\left(1+\left(2-\frac{\beta^2}{4\pi}\right)dl\right)}{(2\pi a)^2}\int_a d^2x' \cos\left[\beta\left(1-\frac{A}{2}g^2\beta^2 dl\right)\phi'_\Lambda\right]$$

introducing $\Delta = \frac{\beta^2}{4\pi}$

$$\Rightarrow \frac{dg}{dl} = (2-\Delta)\,g$$

some const

$$\frac{d\left(\beta^2/4\pi - 2\right)}{dl} = -\frac{2\beta}{4\pi}\frac{d\beta}{dl} = -\frac{\beta^4}{4\pi}Ag^2 \simeq -Ag^2$$

$$\boxed{\begin{cases} \dfrac{dg}{dl} = -(\Delta-2)\, g \\[2ex] \dfrac{d(\Delta-2)}{dl} \simeq -A g^2 \end{cases}}$$

here $\Delta = \dfrac{16\pi K}{4\pi} = 4K$

$$\Rightarrow A\frac{g\,dg}{dl} - \frac{(\Delta-2)\,d(\Delta-2)}{dl} = 0 \Rightarrow g^2 - \frac{(\Delta-2)^2}{A} = 0$$