Lect 2: edge modes of spin chains, AKLT, String order

① Schwinger boson representation of SU(2) algebra

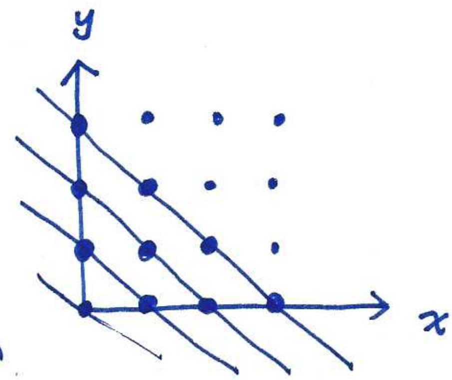
Consider a 2D harmonic oscillator $H = \hbar\omega(a^\dagger a + b^\dagger b)$, where

$a = \frac{1}{\sqrt{2}}[\frac{x}{l} + i\frac{lp_x}{\hbar}]$, $b = \frac{1}{\sqrt{2}}(\frac{y}{l} + i\frac{lp_y}{\hbar}]$ are anhilation opearators.

Apparently. H has an SU(2) symmetry, since $H = \hbar\omega(a^\dagger\ b^\dagger)\begin{pmatrix} a \\ b \end{pmatrix}$. It is invariant under the transformation $\begin{pmatrix} a \\ b \end{pmatrix} \to u\begin{pmatrix} a \\ b \end{pmatrix}$ where $u^\dagger u = 1$.

The energy level degeneracy

$$\boxed{n_a + n_b = 2S}.$$



where $2S$ is an integer. The $2S+1$ fold degenerate states form a representation of the SU(2) group — then we can use this to represent spin in terms of simple operators. Certainly, the price is that a and b are constrainted!

$$\to \quad S_x = \frac{1}{2}(a^\dagger b + b^\dagger a), \quad S_y = \frac{1}{2i}(a^\dagger b - b^\dagger a), \quad S_z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$

or $\vec{S} = \frac{1}{2}(a^\dagger\ b^\dagger)\,\vec{\sigma}\begin{pmatrix} a \\ b \end{pmatrix}$ under the constraint $\frac{1}{2}(n_a + n_b) = 2S$

HW: Verify that the above define $\vec{S}$ satisfies the SU(2) algebra

$$[S_i, S_j] = i\,\epsilon_{ijk} S_k.$$

② Jordan - Wigner transformation for spin-1/2.

We can map spin ↑ as particle and spin ↓ as vaccum state. Then each site can at most be occupied by one particle, and particles at different sites commute. — spin-1/2 is equivalent to hard core boson.

Hard core boson is different from fermion. It only exhibits exclusion in real space, but not in momentum space. In fact, hard core bosons can develop BEC at 2 and 3-dimensions. Nevertheless, in 1D, there exists a non-local transformation changing hard core boson to fermion. The deep reason is the 1D geometry. We denfine

$$\begin{cases} c_i^+ = S_+(i)\, e^{i\pi \sum\limits_{j<i}(S_z(j)+1/2)}, \quad c_i = S_-(i)\, e^{-i\pi \sum\limits_{j<i}(S_z(j)+1/2)} \\ c_i^+ c_i - \frac{1}{2} = S_z(i), \end{cases}$$

▷ The operator in the exponetial is often called string operator.

$$e^{i\pi \sum_{j<i} (S_z(j)+1/2)} = \prod_{j<i} (-)^{(S_z(j)+1/2)}$$

It counts the even/oddness of the occupied sites (spin ↑) to the left of site i.

$\boxed{\text{HW: Verify the above defined } c_i, c_i^\dagger \text{ satisfying the fermion algebra, i.e. } \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0,\ \{c_i\, c_j^\dagger\} = \delta_{ij}.}$

HW: Time-reversal transformation on Jordan-Wigner fermions

Solution: $T\vec{S}T^{-1} = -\vec{S}$ and $T\,i\,T^{-1} = -i$

$$c_i^\dagger = S_+(i) \prod_{j<i} (-)^{S_z(j)+1/2} \Rightarrow T c_i^\dagger T^{-1} = \left(-S_x(i) + i S_y(i)\right) \prod_{j<i} (-)^{-S_z(j)+1/2}$$

$$\Rightarrow T c_i^\dagger T^{-1} = -S_-(i) \prod_{j<i} \left[(-)^{-1}\right]^{S_z(j)+1/2} (-) = (-)^i S_-(i) \prod_{j<i} (-)^{S_z(j)+1/2}$$

$$= (-)^i c_i$$

$$T c_i T^{-1} = -S_+(i) \prod_{j<i} (-)^{S_z(j)+1/2} = (-)^i c_i^\dagger$$

$$\boxed{\begin{aligned} T c_i^\dagger T^{-1} &= (-)^i c_i \\ T c_i T^{-1} &= (-)^i c_i^\dagger \end{aligned}}$$

$\Rightarrow$ After J-W transf. $H = J \sum S_i \cdot S_j \longrightarrow H = \sum_i J (c_i^\dagger c_{i+1} + h.c) + J_z (c_i^\dagger c_i - 1/2)(c_{i+1}^\dagger c_{i+1} - \ldots$

→ The fermion version

$$T H T^{-1} = \sum_i J (-)^i (-)^{i+1} c_i c_{i+1}^{+} + h.c + J_z \sum_i (c_i^{+} c_i - 1/2)(c_{i+1}^{+} c_{i+1} - 1/2)$$

$$= J \sum_i \left(c_i^{+} c_{i+1} + h.c \right) + \cdots = H$$

The stagger sgn is nessary!

③ By drawing an analogy with the SSH model, we can define an spin-1/2 version.

$$H = \sum_i J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_{i+1} \cdot \vec{S}_{i+2}$$

spin-1/2 moment at the bondary

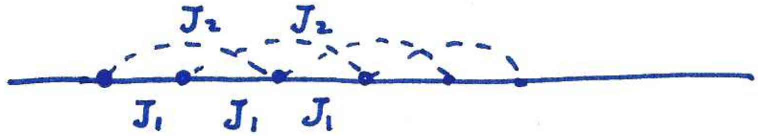
if 

a spin 1/2 moment at the kink

HW: please prove these pictures of edge and domain spin-1/2 moments

④ The above picture is over-simplified. Unless $J_1 = 0$, the spin-1/2 moment is not completely localized on a single site, but has a finit distributi
It would be intersting to figure out the wavefunction in the S_z-basis.

Majumdar - Ghosh model — spontanous dimerization



$$H = J_1 \sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1} + J_2 \sum_j \vec{S}_j \cdot \vec{S}_{j+2}$$ for spin-1/2

Exercise: If $J_2 = J_1/2 \Rightarrow H = J_2 \sum_j (\vec{S}_j + \vec{S}_{j+1} + \vec{S}_{j+2})^2$ - const

For 3 sites, the total spin can be either 1/2 or 3/2,

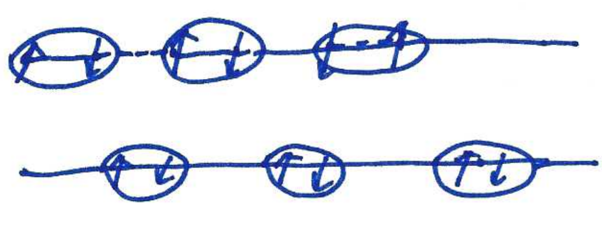
$$\Rightarrow H = J_2 \sum_j \frac{15}{4} P_{3/2}(j, j+1, j+2) + \frac{3}{4} P_{1/2}(j, j+1, j+2)$$

P is the projection operator, since $P_{3/2} + P_{1/2} = 1 \Rightarrow$

$$\boxed{H = 3J_2 \sum_j P_{3/2}(j, j+1, j+2) + \text{const}}$$

If we can minimize $P_{3/2}(j, j+1, j+2)$ for all j's, then the ground state is achieve.

HW: ① Prove that the dimerize state is a ground state.



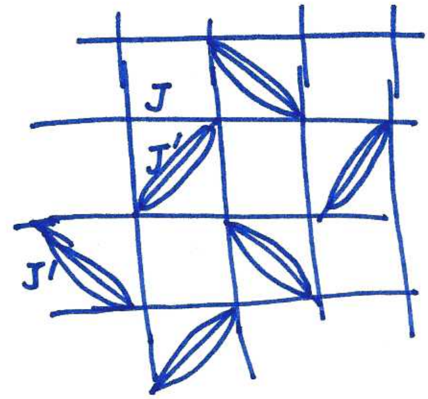
Spontanously breaking of translation symmetry.

where  means 2-sites form a singlet!

② ~~Fractional~~: Fractionalized excitations - spin 1/2

Another dimerized state is the 2D Shastry-Sutherland model

This lattice is based on the square lattice, with some diagonal links shown in the figure.



$$H = J \sum_{\langle ij \rangle} \vec{S}_i \vec{S}_j + J' \sum_{\langle\langle ij \rangle\rangle} \vec{S}_i \cdot \vec{S}_j$$

links on the square   diag links

HW: ① Prove that the dimer states depicted above is actually an eigenstate of H.

② Can you figure out when this dimerized state is actually a ground state?

③ Search in the literature, find the materials with dimerized ground states which can be described by this model.

④ Spin-1 models (1D) – AKLT

$$H_{AKLT} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + 1/3 \left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + 2/3$$

$$\vec{S}_i \cdot \vec{S}_{i+1} = \frac{1}{2}\left(\vec{S}_i + \vec{S}_{i+1}\right)^2 - 2 = \begin{cases} 1 & \text{for } S_{tot} = 2 \\ -1 & = 1 \\ -2 & 0 \end{cases}$$

$$\rightarrow \quad \vec{S}_i \cdot \vec{S}_{i+1} = P_2(i,i+1) - P_1(i,i+1) - 2P_0(i,i+1)$$

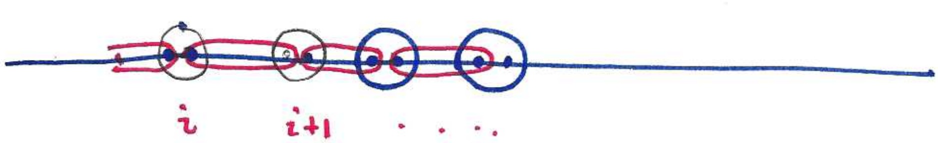
$$\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 = P_2(i,i+1) + P_1(i,i+1) + 4P_0(i,i+1)$$

with $\quad P_2(i,i+1) + P_1(i,i+1) + P_0(i,i+1) = 1$

$$\Rightarrow \boxed{H_{AKLT} = J \sum_i P_2(i, i+1)}$$

For spin-1, it can written as 2-Schwinger bosons

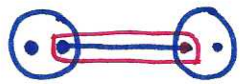
$$\begin{array}{lll} S_z = 1 & \frac{1}{\sqrt{2}}(a^+)^2 |0\rangle & \\ 0 & a^+ b^+ |0\rangle & a^+a + b^+b = 2 \\ -1 & \frac{1}{\sqrt{2}}(b^+)^2 |0\rangle & \end{array}$$



Consider state $|\psi\rangle = \prod_i \left(a_i^+ b_{i+1}^+ - b_i^+ a_{i+1}^+\right)|0\rangle$

Then $|\psi\rangle$ is a ground state.

Consider a bond i, i+1. If we trace out the states of all other sites, we are left the configuration



$$(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) \otimes \text{[one boson on } i \text{ and one boson on } i+1 \text{]}$$

$$a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger = (a_i^\dagger \;\; b_i^\dagger) \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{\text{charge conjugation matrix } R} \begin{pmatrix} a_{i+1}^\dagger \\ b_{i+1}^\dagger \end{pmatrix}$$

$R\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ under rotation, transforms the same as $\begin{pmatrix} a \\ b \end{pmatrix}$

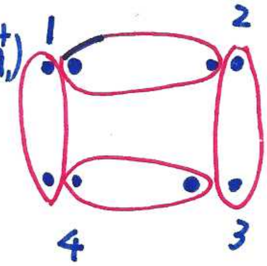
Excercise: prove that actually $(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)$ is invariant under rotation, i.e. form singlet. $\Rightarrow$ the total spin of sites i, i+1 only comes from one boson on "i" and one boson on "i+1". The total spin can only be 0 or 1. $\Rightarrow$ $\boxed{P_2(i, i+1)\, |\psi\rangle = 0.}$

$\Rightarrow$ Since P_2 is a non-negative operator, $\Rightarrow$ $|\psi\rangle$ is a ground state of H.

(I do not know how to prove the uniqueness of the ground state. I guess it is also true).

Let me use a 4-site ring to illustrate the WF

$$|\psi\rangle = (a_1^+ b_2^+ - b_1^+ a_2^+)(a_2^+ b_3^+ - b_2^+ a_3^+)(a_3^+ b_4^+ - b_3^+ a_4^+)(a_4^+ b_1^+ - b_4^+ a_1^+)$$



$$= a_1^+ b_2^+ a_2^+ b_3^+ a_3^+ b_4^+ a_4^+ b_1^+ - a_1^+ b_2^+ a_2^+ b_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

L L L L   L L L R

$$- a_1^+ b_2^+ a_2^+ b_3^+ b_3^+ a_4^+ a_4^+ b_1^+ + a_1^+ b_2^+ a_2^+ b_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

L L R L   L L R R

$$- a_1^+ b_2^+ b_2^+ a_3^+ a_3^+ b_4^+ a_4^+ b_1^+ + a_1^+ b_2^+ b_2^+ a_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

L R L L   L R L R

$$+ a_1^+ b_2^+ b_2^+ a_3^+ b_3^+ a_4^+ a_4^+ b_1^+ - a_1^+ b_2^+ b_2^+ a_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

L R R L   L R R R

$$- b_1^+ a_2^+ a_2^+ b_3^+ a_3^+ b_4^+ a_4^+ b_1^+ + b_1^+ a_2^+ a_2^+ b_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

R L L L   R L L R

$$+ b_1^+ a_2^+ a_2^+ b_3^+ b_3^+ a_4^+ a_4^+ b_1^+ - b_1^+ a_2^+ a_2^+ b_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

R L R L   R L R R

$$+ b_1^+ a_2^+ b_2^+ a_3^+ a_3^+ b_4^+ a_4^+ b_1^+ - b_1^+ a_2^+ b_2^+ a_3^+ \cancel{b_3^+ a_4^+}\ a_3^+ b_4^+ \ b_4^+ a_1^+$$

R R L L   R R L R

$$- b_1^+ a_2^+ b_2^+ a_3^+ b_3^+ a_4^+ a_4^+ b_1^+ + b_1^+ a_2^+ b_2^+ a_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

R R R L   R R R R

$$= 2|0000\rangle - \tfrac{1}{2}|100\text{-}1\rangle - \tfrac{1}{2}|00\text{-}11\rangle + \tfrac{1}{2}|10\text{-}10\rangle$$

$$- \tfrac{1}{2}|0\text{-}110\rangle + \tfrac{1}{4}|1\text{-}11\text{-}1\rangle + \tfrac{1}{2}|0\text{-}101\rangle - \tfrac{1}{2}|1\text{-}100\rangle$$

$$- \tfrac{1}{2}|\text{-}1100\rangle + \tfrac{1}{2}|010\text{-}1\rangle + \tfrac{1}{4}|\text{-}11\text{-}11\rangle - \tfrac{1}{2}|01\text{-}10\rangle$$

$$+ \tfrac{1}{2}|\text{-}1010\rangle - \tfrac{1}{2}|001\text{-}1\rangle - \tfrac{1}{2}|\text{-}1001\rangle$$

Pattern observed ① If we remove "0", ⇒ then we have

1 -1 or 1-1 1-1, or a Neel order.

The AKLT ground state can be viewed by adding zeros into the Neel configure.

② states with more "1" and "-1" have less weight

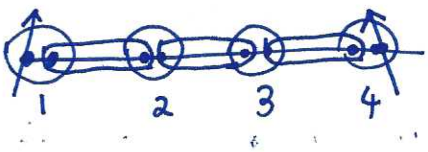
HW ① For the long chain, prove that for the ground state, in the S_z-representation, there does not exist two 1's or two -1's adjacent after 0's are removed.

② $$O_{string} = \lim_{|i-j|\to\infty} \langle S_i^\alpha \, e^{i\pi \sum_{k=i}^{j-1} S_k^\alpha} S_j^\alpha \rangle \qquad \alpha = x, y, z$$

~~This a non-local order develop nonzero expectation value~~.

$O_{string} \neq 0 \rightarrow$ non-local order

⑤ fractional edge excitations — open boundary condition



Two unpaired Schwinger bosons appearing on the edges. How to express in S_z-basis.

$$|\psi\rangle = a_1^\dagger (a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger)(a_2^\dagger b_3^\dagger - b_2^\dagger a_3^\dagger)(a_3^\dagger b_4^\dagger - b_3^\dagger a_4^\dagger)\, a_4^\dagger \, |0\rangle$$

$$= \Big(a_1^\dagger \underset{L}{a_1^\dagger b_2^\dagger}\; \underset{L}{a_2^\dagger b_3^\dagger}\; \underset{L}{a_3^\dagger b_4^\dagger}\; a_4^\dagger \;-\; a_1^\dagger (\underset{L}{a_1^\dagger b_2^\dagger}\; \underset{L}{a_2^\dagger b_3^\dagger}\; \underset{R}{b_3^\dagger a_4^\dagger})\, a_4^\dagger$$

$$- a_1^\dagger \underset{L}{a_1^\dagger b_2^\dagger}\; \underset{R}{b_2^\dagger a_3^\dagger}\; \underset{L}{a_3^\dagger b_4^\dagger}\; a_4^\dagger \;+\; a_1^\dagger \underset{L}{a_1^\dagger b_2^\dagger}\; \underset{R}{b_2^\dagger a_3^\dagger}\; \underset{R}{b_3^\dagger a_4^\dagger}\; a_4^\dagger$$

$$- a_1^\dagger \underset{R}{b_1^\dagger a_2^\dagger}\; \underset{L}{a_2^\dagger b_3^\dagger}\; \underset{L}{a_3^\dagger b_4^\dagger}\; a_4^\dagger \;+\; a_1^\dagger (\underset{R}{b_1^\dagger a_2^\dagger})\; \underset{L}{a_2^\dagger b_3^\dagger}\; \underset{R}{b_3^\dagger a_4^\dagger}\; a_4^\dagger$$

$$+ a_1^\dagger \underset{R}{b_1^\dagger a_2^\dagger}\; \underset{R}{b_2^\dagger a_3^\dagger}\; \underset{L}{a_3^\dagger b_4^\dagger}\; a_4^\dagger \;-\; a_1^\dagger \underset{R}{b_1^\dagger a_2^\dagger}\; \underset{R}{b_2^\dagger a_3^\dagger}\; \underset{R}{b_3^\dagger a_4^\dagger}\; a_4^\dagger \Big) |0\rangle$$

$$= \frac{1}{\sqrt{2}} |1\,0\,0\,0\rangle - \frac{1}{\sqrt{8}} |1\,0\,{-1}\,1\rangle - \frac{1}{\sqrt{2}} |1\,{-1}\,1\,0\rangle + \frac{1}{2} |1\,{-1}\,0\,1\rangle$$

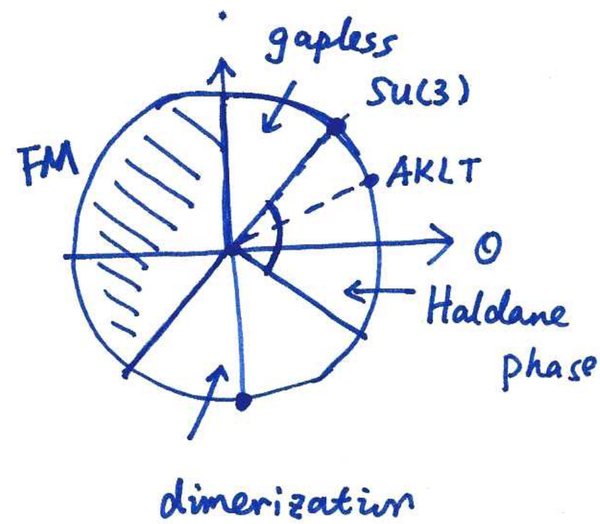
$$- \frac{1}{\sqrt{2}} |0\,1\,0\,0\rangle + \frac{1}{\sqrt{8}} |0\,1\,{-1}\,1\rangle + \frac{1}{\sqrt{2}} |0\,0\,1\,0\rangle - \frac{1}{\sqrt{2}} |0\,0\,0\,1\rangle$$

§ ~~AKLT state~~ 9

Spin-1 mode $H = J \sum_i \cos\theta \, \vec{S}_i \cdot \vec{S}_{i+1} + \sin\theta \, (\vec{S}_i \cdot \vec{S}_{i+1})^2$

~~Sin~~ $\theta = 0$. the pure Heisenberg model lies in the same phase of the AKLT one

	$S_i \cdot S_j$	$(S_i \cdot S_j)^2$	
singlet 0	-2	4	$-2\cos\theta + 4\sin\theta$
triplet 1	-1	1	$-\cos\theta + \sin\theta$
quintet 2	1	1	$\cos\theta + \sin\theta$



① at $\theta = 45^\circ \Rightarrow$ $E_{singlet} = E_{quint}$

SU(3) symmetry $3 \times 3 = 3 + 6$

② at ~~$\theta = 45^\circ$~~ $\theta = -90^\circ$, $E_{triple} = E_{quint}$ $\square \times \square^* = 1 + 8$

SU(3), symmety

§ other AKLT phase