Lect 1 Domain wall fermion and SSH model

§ Jackiw - Rebbi mode — Dirac equation

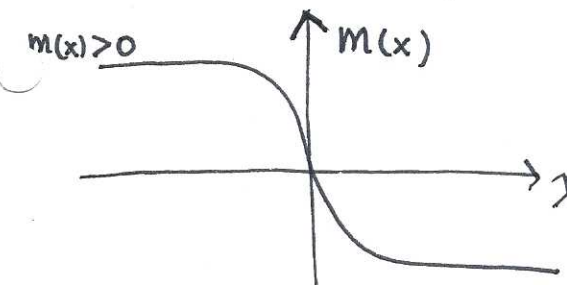
$$H = \vec{\alpha}\cdot\vec{p} + m\beta \quad \Rightarrow \quad E = \sqrt{p^2+m^2} \quad \Rightarrow \text{ spectrum independent of the sign of } m.$$

⇒ If we change the sign of $m \to -m$, it won't cause physical observable effect. However, if m is positive in one part of the system, and negative in the rest. The spectrum will develop zero energy mode localized around the domain. For symplicity, we use 1D case as an illustration.

$$\boxed{H = \alpha p_x + m(x)\beta, \quad \text{where } \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

The diagonal basis for α is called the chiral base, the mass term mixes together the left and right movers. The eigen equation reads

$$\begin{cases} -i\partial_x \psi_1(x) + m(x)\psi_2(x) = E\psi_1(x) \\ m(x)\psi_1(x) + i\partial_x \psi_2(x) = E\psi_2(x) \end{cases} \quad \text{with } H\psi(x) = E\psi(x) \text{ and } \psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$



We seek the solution of zero energy, i.e. $E = 0$. ← "chiral symmetry"

A trick is to find an operator O to anticamute H, i.e. $OH + HO = 0$

The for each eigenstate $H\psi = E\psi \Rightarrow H(O\psi) = -OH\psi = -E(O\psi)$

$\Rightarrow$ $O\psi$ is also an eigenstate with energy $-E$. But for the zero energy state $\Rightarrow$ ψ and $O\psi$ are both in the zero energy sector. We can organize the zero energy sector according to the eigenvector of "O".

Apparently, O can be choosen as $\boxed{\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$, and the zero energy state ψ_0 is an eigenstate of σ_2. $\Rightarrow$ $\boxed{\psi_2 = \pm i\psi_1}$.

It's also easy to directly check

$$\begin{cases} -i\partial_x \psi_1 \pm i m(x)\psi_1 = 0 & ① \\ m(x)\psi_1 \mp \partial_x \psi_1 = 0 & ② \end{cases}$$

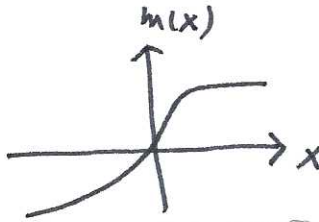
$\Rightarrow$ ① and ② are equivalent to each other, then we only need to solve one.

From ① $\Rightarrow$ $\partial_x \psi_1 = \pm m(x)\psi_1$

$\Rightarrow$ $\psi_1(x) = \psi_1(x=0)\, e^{\pm \int_0^x m(x)dx}$ we want bound states

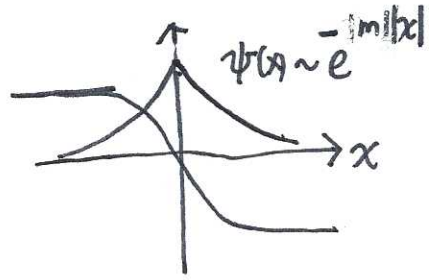
i.e. $\psi(x \to \pm\infty) \to 0$

$$\Rightarrow \begin{cases} \psi_1(x) = \psi_1(x=0)\, e^{\int_0^x m(x)dx} \\ \psi_2(x) = i\psi_1 \end{cases}$$

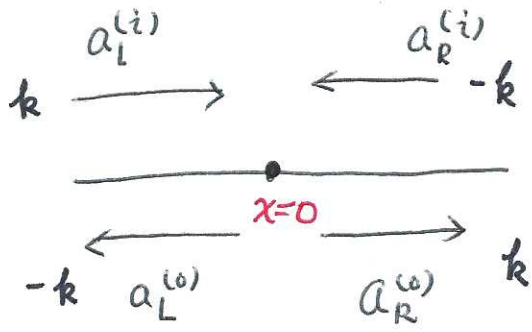


The other solution with $\psi_2(x) = -i\psi_1(x)$ corresponds to

If $m(x) = -m\, \mathrm{sgn}(x) \Rightarrow \begin{cases} \psi_1(x) = \psi_1(x=0)\, e^{-m|x|} \\ \psi_2(x) = i\psi_1 \end{cases} \Rightarrow$



HW 1: So far, we have consider the bound state. Let us solve the scattering states. Write down the



Solution

$$\psi = \begin{cases} a_L^{(i)}\, \psi_{k,E}^{(m>0)} + a_L^{(o)}\, \psi_{-k,E}^{(m>0)} \\ a_R^{(i)}\, \psi_{-k,E}^{(m<0)} + a_R^{(o)}\, \psi_{k,E}^{(m<0)} \end{cases}$$

and write down $\begin{bmatrix} a_L^{(o)} \\ a_R^{(o)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} a_R^{(i)} \\ a_L^{(i)} \end{bmatrix}$.

For simplicity, choose $m(x) = \begin{cases} m, & x<0 \\ -m & x>0, \end{cases}$ $\psi_{k,E}^{(m)}$ and $\psi_{-k,E}^{(m)}$ are plane wave solutions for a uniform mass m.

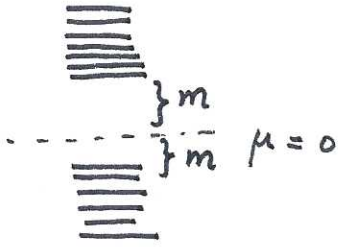
Solve t and r as function k.

Can you obtain bound state through $t(k)$ and $r(k)$?

Open question: Can we develop a Levinson-theorem like result to relate the zero mode with the scattering phase shift? Compared to the bound states in non-relativistic systems, what's special of the phase-shift pattern here?

§ Half-fermion

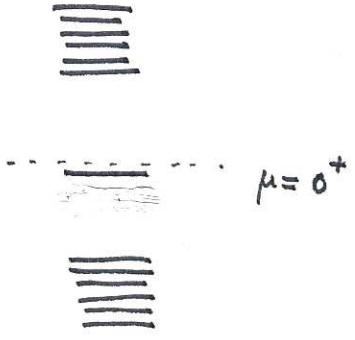
For the case with uniform mass, the spectrum is symmetric with respect to $E=0$ with a gap $2m$. If the chemical potential $\mu=0$, then all the state $E<0$ is occupied, Let us denote the background charge (vacuum) as



$$N_{BG} = \sum_{\substack{\text{Kink}\\ \text{free}}} n(E<0).$$

In the presence of kink, not only there appears a zero mode, but also the vaccum (the occupied scattering states are also occupied).

$$N_{occ} = \sum_{\text{kink}} n(E<0) + 1_{\uparrow E=0}$$



We assum that the total # of states are the same indepedent of the presence of the kink or not.

$$\Rightarrow \sum_{\text{kink free}} n(E<0) + n(E>0) = \sum_{\text{kink}} \left[n(E<0) + n(E>0) \right] + 1$$

Due to the "chiral sym" $\Rightarrow \sum_{\text{kink free}} 2n(E<0) = \left(2 \sum_{\text{kink}} n(E<0) \right) + 1$

$$\Rightarrow N_{occ} - N_{BG} = \left(\sum_{\text{kink}} n(E<0) + 1 \right) - \sum_{\text{kink free}} n(E<0)$$

$$= 1 - 1/2 = \cancel{n_{E=0} - 1/2 n_{E=0} = 1/2 n_{E=0}} = 1/2$$

↖ contribution from the change of vaccum !

We can easily write down the local version

$$\boxed{\delta\rho(x) = \frac{1}{2} \psi_0^\dagger(x) \psi_0(x)}$$

Again it can be explained as

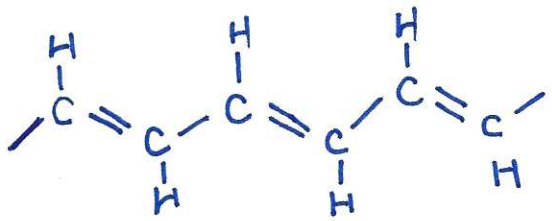
$$\delta\rho(x) = \sum_{\text{kink}} \left\{ \left[\psi^\dagger_{E_n<0}(x) \psi_{E_n<0}(x) \right] + \psi_0^\dagger(x) \psi_0(x) \right\} - \sum_{\text{kink free}} \psi'^\dagger_{E_n<0}(x) \psi'_{E_n<0}(x)$$

we have $2 \sum_{\text{kink}} \psi^\dagger_{E_n<0}(x) \psi_{E_n<0}(x) + \psi_0^\dagger(x) \psi_0(x) = 2 \sum_{\text{kink free}} \psi'^\dagger_{E_n<0}(x) \psi'_{E_n<0}(x)$

$$\Rightarrow \delta\rho(x) = \underset{\uparrow}{\psi_0^\dagger(x) \psi_0(x)} - \frac{1}{2} \underset{\uparrow}{\psi_0^\dagger(x) \psi_0(x)}$$

zero modes — background (vaccum change)

§ Conducting polymers : polyacetylene

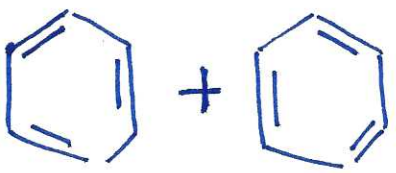


sp^2 - hybridization σ-bond

P_z - electron active, π-bonding

6 π-electrons.

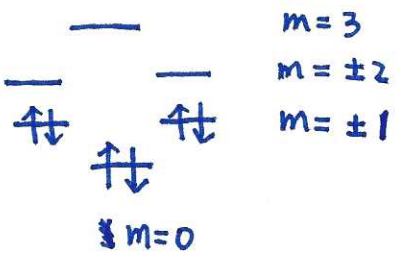
Simple example - benzene



(real space)

Excercise: prove it

=



(momentum space)

✡ Pauling's Resonating valence bond ⟶ Anderson's RVB.

add interaction

project out doubly occupied state

Now let's neglect electron - electron interaction, but keep electron lattice interaction.

$$H = -\sum_{\langle ij \rangle} \left[t_0 + \alpha (l_{ij} - l_0) \right] C_i^\dagger C_j + h.c + \sum_{\langle ij \rangle} \frac{1}{2} K (l_{ij} - l_0)^2$$

↙

SSH model

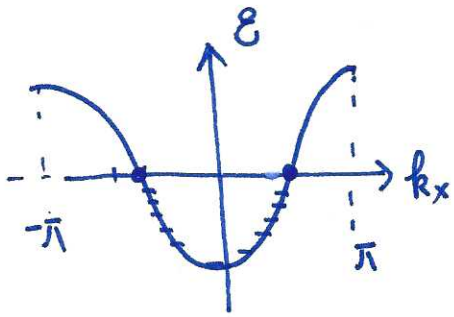
{ l_{ij} denotes the bond strength

l_0 is the bond strength before deformation.

For simplicity, we consider spinless fermion

① If no-dimerization, i.e. $l_{ij} \equiv l_0 \Rightarrow$ uniform tight-binding model

$$H \to H_0 = -t_0 \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c) \Rightarrow \varepsilon_k = -2t \cos k_x$$



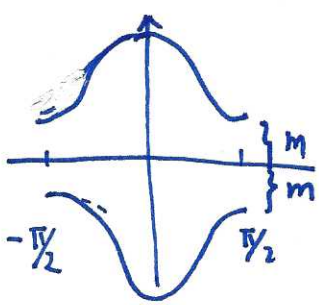
half-filling $\Rightarrow k_f = \pi/2$

$$\frac{E_0}{V} = -\int_{-\pi/2}^{\pi/2} \frac{dk}{2\pi}\, 2t \cos k = -\frac{2t}{2\pi} \sin k \Big|_{-\pi/2}^{\pi/2} = -\frac{2t}{\pi}$$

② dimerization $l_{ij} = l_0 + (-)^i \Delta l$ and define $m = \alpha \Delta l$

$$H = -\sum_{\langle ij \rangle} (t_0 + (-)^i m)\, c_i^\dagger c_j + h.c + \sum_{\langle ij \rangle} \frac{1}{2} K (\Delta l)^2$$

Ex: prove the spectrum $E_{\pm k} = \pm \sqrt{(2t \cos k_x)^2 + m^2}$



— = — = — = — = unit cell doubled

BZ → $[-\pi/2, \pi/2]$

2-bands, gap opens

Peierls transition

insulator →

$$\frac{E_0'}{V} = -\int_{-\pi/2}^{\pi/2} \frac{dk}{2\pi}\, 2t \sqrt{(\cos k_x)^2 + \left(\frac{m}{2t}\right)^2} \approx -2t \left[1 + \frac{m^2}{(2t)^2}\right]^{1/2} \cdot \int_{-\pi/2}^{\pi/2} \frac{dk}{2\pi} \cdot$$

$$\left(1 + \left(1 - \frac{m^2}{(2t)^2}\right) \sin^2 k\right]^{1/2}$$

According to elliptic integral $\int_{-\pi/2}^{\pi/2} dk\,(1-(1-\alpha^2)\sin^2 k)^{1/2} \simeq$

$$\sim 2 + (a_1 - b_1 \ln \alpha^2)\,\alpha^2$$

(a_1, b_1 consts)

$\Rightarrow$ the leading order energy (electronic) change

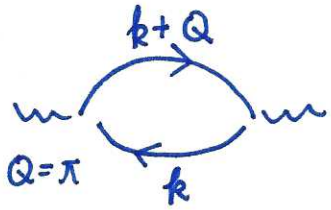
$$\frac{E_0'}{V} - \frac{E_0}{V} \approx -\frac{2t}{\pi} b_1 \left[\ln \frac{(2t)^2}{m^2}\right] \cdot m^2 + O(m^2)$$

while the kinetic energy increase $\frac{E_{elastic}}{V} = \frac{1}{2} K \left(\frac{m}{\alpha}\right)^2$.

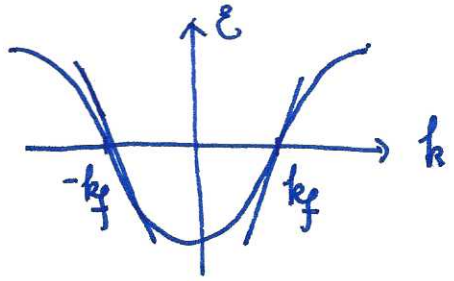
$\Rightarrow$ The electron energy gain is always larger than lattic energy cost $\rightarrow$ 1D always has Peirerls instability.

or the 1D electron systems has divergent susceptibility with-respect to dimerization $-\frac{d^2}{dm^2}\frac{\Delta E_0}{V} \approx \frac{2t}{\pi} b_1 \ln\left(\frac{(2t)^2}{m^2}\right)\Big|_{m\to 0} \rightarrow \infty$.

or ~~HW~~ HW: From Lindhard response, calculate the $\chi_0(\pi, \omega)$ and set $\omega \to 0$. Find the logrithmic divergence. This divergence arises from the Fermi surface nesting.



§ Map to relativistic electrons

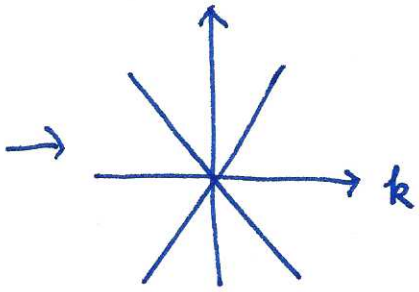


linearize the spectrum around $\pm k_f$

→ right and left movers

$$\frac{1}{\sqrt{a}} C_i \sim \psi_R(x)\, e^{i k_f \cdot R_i} + \psi_L(x)\, e^{-i k_f \cdot R_i}$$

a: lattice constant



$$\psi_{R,L}(x) = \sum_{|k|<\Lambda} \psi_{R,L}(k)\, e^{ikx} ,$$

Λ is a cut off

HW: Prove that after the linerization

$$H = \int dx \; (\psi_R^\dagger(x) \;\; \psi_L^\dagger(x)) \left[v \begin{bmatrix} -i\partial_x & 0 \\ 0 & i\partial_x \end{bmatrix} + m \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}$$

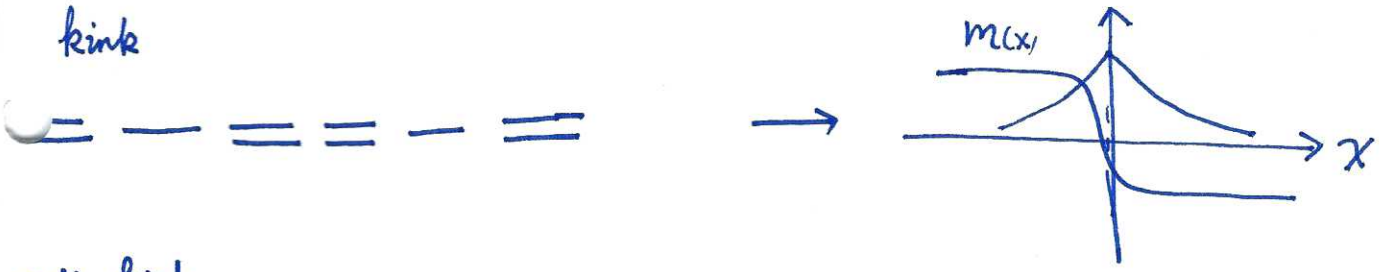
where $v = 2t$ is the Fermi velocity, m is the gap.

Prove

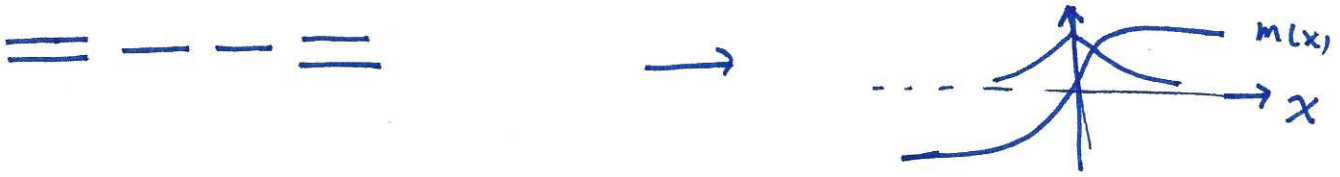
— = — = — = — = these 2 configurations

= — = — = — = — correspond to $\pm m$.

kink

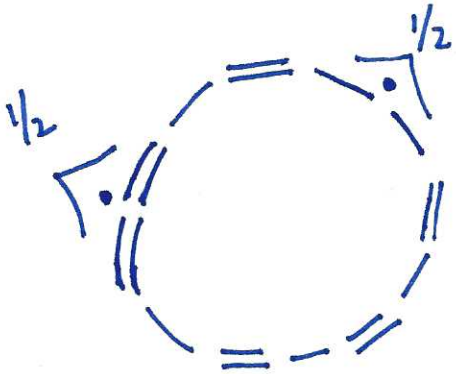


anti-kink

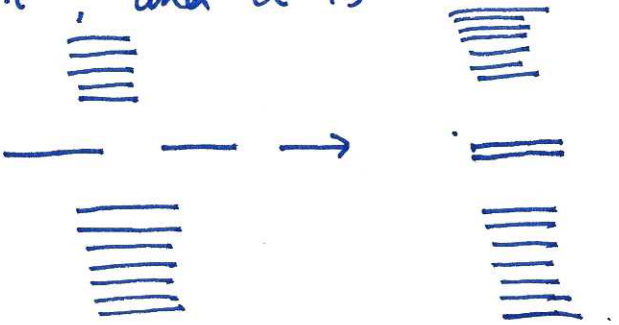


For periodical boundary condition, and even number of sites.

kink and anti-kink must appear in pairs. The zero modes must appear in pairs.



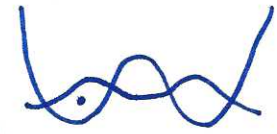
If the length of the system is finite, then these two modes are not exactly at zero energy, but hybridize and has a small splitting $\Delta E \sim e^{-d/\xi}$ where $\xi = m^{-1}$, and "d" is the distance between kink-antikink.



Imagine $\mu \to 0^+$, that both zero modes are occupied, then compared to the kink-free background (the vaccum), the system has one

more electron. But the density distribution splits into $1/2 + 1/2$ and can be seperated as far as you want $\longrightarrow$ fractionalization!

HW: Figure out the difference between fractionalization and superposition. Say, a double-well problem,



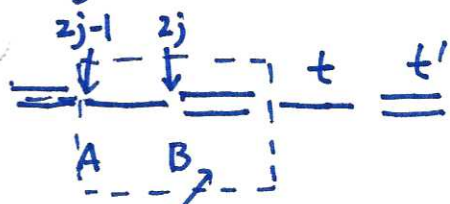
the probility of an electron in the left/right well is $1/2$. The key is fluctuation: In the double well case, the L-well is not the eigenstate, there're particle number fluctuation

$50\% \times 0 + 50\% \cdot 1 = 1/2$. $\longrightarrow$ variation $1/2 - (1/2)^2 = 1/4$.

But for the case of fractionalization, please check that within in the length scale of one zero mode, the particle number fluctuation $\longrightarrow 0$, as the system size $L \to \infty$. Although fractionalization here is based on non-interacting system, it still needs many electrons. just (not) a purely single electron property.

please do a ring with 100 or 200 sites, and solve it with exact diagonalization. Then try to prove it analytically. Fractionalized electron is a particle number eigenstate, not just the averge is $1/2$.

§ Index 2(1)



ith unit cell, lattice constant 2a

$$C_{A,k} = \frac{1}{\sqrt{N}} \sum_j C_{2j-1}\, e^{ik(2j-1)a}$$

$$C_{B,k} = \frac{1}{\sqrt{N}} \sum_j C_{2j}\, e^{ik(2j-1)a}$$

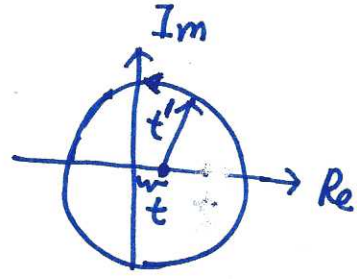
N – # of unit cell

$$\Rightarrow H = \sum_k \left(C^\dagger_{A,k}\ C^\dagger_{B,k}\right) h(k) \begin{pmatrix} C_{A,k} \\ C_{B,k} \end{pmatrix}$$

$t' > t$

$$h(k) = \begin{pmatrix} 0, & t + t' e^{2ika} \\ t + t' e^{-2ika} & 0 \end{pmatrix}$$

The only ~~non-zero~~-independent element is $t + t' e^{2ika}$, and let's plot it in the complex plane. Since period doubling $k \in [-\pi/2, \pi/2]$ or $2ka \in [-\pi, \pi]$. $\Rightarrow$



$h_{11} = t + t' e^{2ika}$ winds around the origin as k runs from $-\pi/2$ to $\pi/2$.

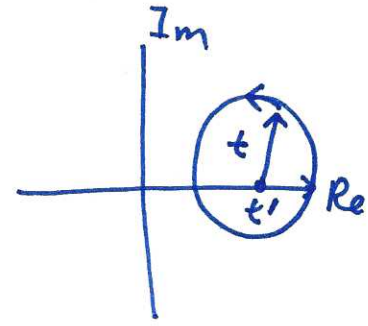
More formally, we can define the winding number.

$$\omega = \frac{1}{2\pi i} \oint dk\ \ h_{11}^{-1}\, \partial_k\, h_{11} = 1$$

Howerver, if we change the choice of the unit cell

it corresponses to switch the role of t and t'.

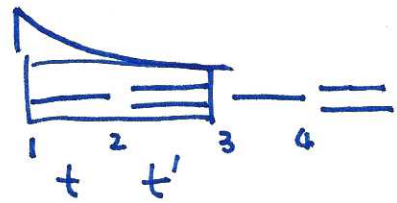
then $h(k) = \begin{pmatrix} 0 & t' + t e^{2ika} \\ c.c. & 0 \end{pmatrix} \Rightarrow$



$$\omega = \frac{1}{2\pi i} \oint dk \; h_{11}^{-1} \partial_k h_{11} = 0$$

what happens? The value of ω depends on how we chose the unit cell. But once the convention of unit cell is fixed, then two different dimer configuratins correspond to two different winding numbers. Only the relative winding number has physical meaning, which gives rise to the number of zero modes when put two different dimer config together to creat kinks.

- Edge modes



If a weak bond terminates at the boundary, then there's a zero energy edge mode localized on the boundary

HW: please use transfer matrix method to solve the edge mode.

$$|\psi\rangle = \sum_{j=1}^{\infty} \left[a_{2j-1} |2j-1\rangle + a_{2j} |2j\rangle \right]$$

$$H|\psi\rangle = -t a_2 |1\rangle + (-t a_1 - t' a_3)|2\rangle + (-t' a_2 - t a_4)|3\rangle + \cdots$$

$$E|\psi\rangle = E a_1 |1\rangle + E a_2 |2\rangle + \cdots$$

$$\Rightarrow \begin{cases} -t a_2 = E a_1 \\ -t a_1 - t' a_3 = E a_2 \\ -t' a_2 - t a_4 = E a_3 \\ -t a_3 - t' a_5 = E a_4 \\ \vdots \quad = \end{cases}$$

Set $E = 0 \Rightarrow a_2 = a_4 = \cdots = 0$

$$\begin{cases} a_3 = -\frac{t}{t'} a_1 \\ a_5 = -\frac{t}{t'} a_3 \\ \vdots \end{cases}$$

$\Rightarrow$ if $t' > t \Rightarrow$ there exist an localized edge mode

$$\begin{cases} a_{2j-1} = (-)^{j-1} \left(\frac{t}{t'}\right)^{j-1} a_1 \\ a_{2j} = 0 \end{cases}$$

relation between

HW: Figure out the bases in which the left and right movers are diagonalized, and the basis in real space used here.