lect 11. Free fermion Green's function

§ time-ordered, imaginary time Green's function

$$C_k(t) = e^{iHt} C_k e^{-iHt}$$

$$i G(t_1-t_2, k_1)\delta_{k_1,k_2} = \langle 0| T(C_{k_1}(t_1) C^\dagger_{k_2}(t_2)|0\rangle = \begin{cases} \langle 0|C_{k_1}(t_1) C^\dagger_{k_2}(t_2)|0\rangle & \text{for } t_1>t_2 \\ -\langle 0|C^\dagger_{k_2}(t_2) C_{k_1}(t_1)|0\rangle & \text{for } t_1<t_2 \end{cases}$$

if at finite temperature $\langle 0|\cdots|0\rangle \to \frac{1}{Z} tr[e^{-\beta H}\ldots]$.

$$i G(t, k) = \begin{cases} \theta(t)\,\theta(\xi_k)\, e^{-it\xi_k} - \theta(-t)\,\theta(-\xi_k)\, e^{-i(-t)(-\xi_k)} \Big|_{T=0\,K} \\ \theta(t)\,(1-n_F(\xi_k))\, e^{-it\xi_k} - \theta(-t)\, n_F(\xi_k)\, e^{-it\xi_k} \Big|_{T>0} \end{cases}$$

$$\Rightarrow \quad G(\omega, k) = \begin{cases} \dfrac{1}{\omega - \xi_k + i0^+ \operatorname{sgn}(\omega)} & \text{for } T = 0\,K \\[2ex] \dfrac{1-n_F(\xi_k)}{\omega - \xi_k + i0^+} + \dfrac{n_F(\xi_k)}{\omega - \xi_k - i0^+} & \text{for } T>0 \end{cases} \qquad \boxed{\xi_k = \varepsilon_k - \mu}$$

$$C_k(\tau) = e^{H\tau} C_k e^{-H\tau}$$

$$\mathcal{G}^\beta(\tau_1, \tau_2; k_1)\,\delta_{k_1,k_2} = -\frac{1}{Z} tr[e^{-\beta H} C_{k_1}(\tau_1) C^\dagger_{k_2}(\tau_2)] \quad (\text{if } \tau_1>\tau_2)$$
$$+ \frac{1}{Z} tr[e^{-\beta H} C^\dagger_{k_2}(\tau_2) C_{k_1}(\tau_1)] \quad \text{if } (\tau_2>\tau_1)$$

for fermions $\mathcal{G}^\beta$ satisfies $\mathcal{G}^\beta(\tau, k) = -\mathcal{G}^\beta(\tau+\beta, k)$

$$\mathcal{G}^{\beta}(\tau, k) = -\Theta(\tau)\,(1-n_F(\xi_k))\,e^{-\tau\xi_k} + \Theta(-\tau)\,n_F(\xi_k)\,e^{-\tau\xi_k}$$

$$\Rightarrow \mathcal{G}^{\beta}(\omega_n, k) = \int_0^{\beta} d\tau\; \mathcal{G}^{\beta}(\tau, k)\,e^{i\omega_n\tau} = \frac{1}{i\omega_n - \xi_k} \qquad \omega_n = \frac{(2n+1)\pi}{\beta}$$

§ Equal-space Green's function and tunneling:

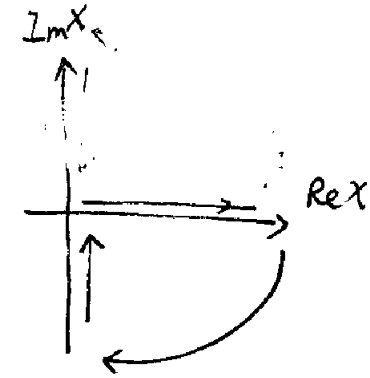
Let's evaluate $\langle T\; c(x_1 t_1)\,c^{\dagger}(x_2 t_2)\rangle$ at $x_1 = x_2$

$$i\,G(t,0) = \int\frac{d^dk}{(2\pi)}\left[\Theta(t)\,\Theta(\xi_k)\,e^{-it\xi_k} - \Theta(-t)\,\theta(-\xi_k)\,e^{-it\xi_k}\right]$$

$$\int\frac{d^dk}{(2\pi)^d} = \int_{\substack{0\\-\infty}}^{+\infty} d\xi\; N(\xi + E_F)\,, \qquad \text{if } N(\xi+E_F) \propto |\xi|^g$$

$$\Rightarrow \Theta(t)\int_0^{+\infty} d\xi\; N(\xi+E_F)\,e^{-it(\xi - i0^+)} =$$

$$\propto \Theta(t)\int_0^{+\infty} d\xi\;\xi^g\,e^{-it(\xi - i0^+)} = \theta(t)\,t^{-(1+g)}\int_0^{+\infty} dx\; x^g\,e^{-i(x-i0^+)}$$



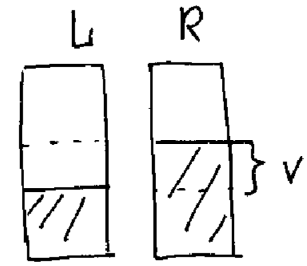
$$\int_0^{+\infty} dx\; x^g\,e^{-ix} + \int_{arc} dz\; z^g\,e^{-i(\mathrm{Re}\,z + i\,\mathrm{Im}\,z)}$$

$$+ \int_{-\infty\cdot i}^{0} dz\; z^g\,e^{-iz} = 0$$

$$\Rightarrow \int_0^{+\infty} dx\; x^g\,e^{-ix} = \int_0^{+\infty} d(-ix)\,(-ix)^g\,e^{-x} = (-i)^{g+1}\,\Gamma(g-1)$$

$$\Rightarrow \; G(t,0) \propto -i\,\mathrm{sgn}(t)\, e^{-i\frac{\pi}{2}(1+g)} \frac{1}{|t|^{1+g}}$$

how to measure this? we can use tunneling spectroscopy.



$$N_{R,L}(\xi) \sim |\xi|^{g_{R,L}}$$

$$H_T = \Gamma \left(e^{iVt} C_R^\dagger C_L + h.c. \right)$$

$$\langle j_T \rangle(t) = -i \int_{-\infty}^{t} dt' \; \langle | [j_T(t), \; H_T(t')] | \rangle$$

$$= -\Gamma^2 \int^t dt' \; e^{iV(t-t')} \langle | [I(t), I^\dagger(t')] | \rangle \quad + h.c., \quad \text{where } I(t) = C_R^\dagger(t,0)\, C_l(t,0)$$

$$= -i\,\Gamma^2 \, D_R(V) + h.c. = 2\Gamma^2 \, \mathrm{Im}\, D_R(V)$$

$$\langle I(t) I^\dagger(0) \rangle = \langle C_L(t,0) C_L^\dagger(00) \rangle \langle C_R^\dagger(t0)\, C_R(00) \rangle = -G_L(t,0)\,(-)\,G_R(t,0) \quad (\text{for } t>0)$$

$$\propto e^{-i\frac{\pi}{2}(2+g_R+g_L)} \frac{1}{t^{2+g_R+g_L}}$$

$$\cancel{\langle j_T \rangle = -\Gamma^2 \int_{-\infty}^{t} dt' \; e^{iV(t-t')} \; (t-t')^{-(2+g_R+g_L)} \; e^{-i\frac{\pi}{2}}}$$

$$\langle I^\dagger(0)\, I(t) \rangle = \langle C_L^\dagger(00)\, C_L(t,0) \rangle \langle C_R(00)\, C_R^\dagger(t,0) \rangle = G_L^*(t,0)\, G_R^*(-t,0) \quad (\text{for } t>0)$$

$$\Rightarrow \langle | [I(t), I^\dagger(t')] | \rangle \propto \left(e^{-i\frac{\pi}{2}(2+g_R+g_L)} - h.c. \right) \frac{1}{|t-t'|^{2+g_R+g_L}}$$

$$= -2i \sin \frac{(2+g_R+g_L)\pi}{2} \; \frac{1}{|t-t'|^{2+g_R+g_L}}$$

$$\Rightarrow \langle j_T \rangle (t) = + P^2 \underbrace{\int_{-\infty}^{t} dt' \left(e^{iV(t-t')} \frac{i}{|t-t'|^{2+g_R+g_L}} + h.c. \right)}_{2\sin\frac{(2+g_R+g_L)\pi}{2}}$$

$$\propto V^{+(2+g_R+g_L)-1} = V^{+1+(g_R+g_L)}$$

If both side is metal, we arrive at ohom's law, otherwise the tunneling current is suppressed by a factor $V^{+g_R} \cdot V^{g_L}$.

§ Fermion spectral function

we introduce the spectral function defined as

$$i\,G(t,0) = \int d\omega \; A_+^0(\omega)\, e^{-i\omega t} \quad \text{for } t>0$$

$$\pm \int d\omega \; A_-^0(\omega)\, e^{-i\omega t} \quad \text{for } t<0$$

where $A_+^0(\omega) = \sum_{m,n} \delta(\nu - (\epsilon_m - \epsilon_n)) \langle \psi_n | c(x) | \psi_m \rangle \langle \psi_m | c^\dagger(x) | \psi_n \rangle \frac{e^{-\beta\epsilon_n}}{Z}$

$$A_-^0(\omega) = \sum_{mn} \delta(\nu + (\epsilon_m - \epsilon_n)) \langle \psi_n | c^\dagger(x) | \psi_m \rangle \langle \psi_m | c(x) | \psi_n \rangle \frac{e^{-\beta\epsilon_n}}{Z}$$

Similarly, we can introduce the spectral function ρ in momentum space.

$$i\,G(t,k) = \int d\omega \; A_+^0(\omega,k)\, e^{-i\omega t} \quad \text{for } t>0$$

$$\int d\omega \; A_-^0(\omega,k)\, e^{-i\omega t} \quad \text{for } t<0$$

and $G_+(\omega,k) = \int dt\, \theta(t)\, G(t,k)\, e^{-i\omega t} = \int d\nu \dfrac{A_+(\nu,k)}{\omega - \nu + i0^+}$

$$G_-(\omega,k) = \int dt\, \theta(-t)\, G(t,k)\, e^{-i\omega t} = \int d\nu \frac{A_-(\nu,k)}{\omega - \nu - i0^+}$$

where $A_+(\nu,k) = \sum_{mn} \delta[\nu - (\epsilon_m - \epsilon_n)] \langle \psi_n | C_k | \psi_m \rangle \langle \psi_m | C_k^\dagger | \psi_n \rangle \dfrac{e^{-\beta\epsilon_n}}{Z}$

$$A_-(\nu,k) = \sum_{mn} \delta[\nu + (\epsilon_m - \epsilon_n)] \langle \psi_n | C_k^\dagger | \psi_m \rangle \langle \psi_m | C_k | \psi_n \rangle \frac{e^{-\beta\epsilon_n}}{Z}.$$

Let us rederive the tunneling current using the spectral function.

$$\langle [I(t), I^\dagger(0)] \rangle = \langle C_L(t) C_L^\dagger(0) \rangle \langle C_R^\dagger(t,0) C_R(00) \rangle - \langle C_L^\dagger(00) C_L(t,0) \rangle \langle C_R(00) C_R^\dagger(t0) \rangle$$

$$= (iG_L(t))(-iG_R(-t)) - (-iG_L(-t))^* (iG_R(t))^*$$

$$= -\int d\omega_L\, d\omega_R \left[A^0_{L+}(\omega_L) A^0_{R-}(\omega_R) - A^0_{L-}(\omega_L) A^0_{R+}(\omega_R)\right] e^{i(\omega_R - \omega_L)t}$$

The retarded Green function : $iD(t) = \theta(t) \langle [I(t), I^\dagger(0)] \rangle$

$$D(\omega) = \int dt\, D^\phi(t)\, e^{+i\omega t}$$

$$= \int d\omega_L\, d\omega_R \frac{A^0_{L+}(\omega_L) A^0_{R-}(\omega_R) - A^0_{L-}(\omega_L) A^0_{R+}(\omega_R)}{\omega - (\omega_L - \omega_R) + i0^+}$$

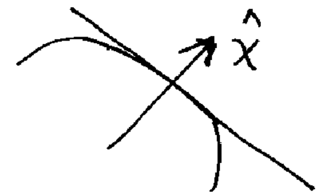
$$\Rightarrow j_T(t) = 2P^2\, \mathrm{Im}\, D(V)$$

$$= 2\pi P^2 \int d\nu \left[(A^0_{L-}(\nu) A^0_{R+}(\nu - V) - A^0_{L+}(\nu) A^0_{R-}(\nu - V) \right]$$

For spherical Fermi surface, $\Rightarrow$ $\quad C_1 = C_2$

$$i\,G(-0^+, x)\Big|_{x\to\infty} \sim \cos\!\left(k_F|x| - \frac{\pi(d+1)}{4}\right)\frac{1}{|x|^{d+1/2}} .$$

If Fermi surface has ~~next~~ flat piece,



then ~~da~~ along that direction

$$N(k, \hat{x}) \simeq \text{const} ,$$

we effective has 1d system ~~sta~~ along $\hat{x}$-direction

$$\to i\,G(-0^+, x)\Big|_{x\to+\infty} \sim -i\, e^{i\vec{k}_F(\hat{x})\cdot\vec{x}}\,\frac{1}{|x|}$$

For free fermions, we have $A_-(\omega,k) = n_F(\xi_k)\,\delta(\omega - \varepsilon_k)$

$$A_+(\omega,k) = (1-n_F(\xi_k))\,\delta(\omega-\varepsilon_k)$$

$$\Rightarrow A_-^0(\omega) = n_F(\omega)\,N(\omega+E_F) \qquad A_+^0(\omega) = \left(1-n_F(\omega)\right) N(\omega+E_F)$$

$$\rightarrow \langle \dot{J}_T\rangle(t) = 2\pi \Gamma^2 \int d\nu \; \Big((1-n_F(\nu))\, n_F(\nu - V)\; N_L(\nu+E_F) N_R(\nu - V + E_F)$$
$$- n_F(\nu)(1-n_F(\nu-V))\Big)\; N_L(\nu+E_F)\,N_R(\nu-V+E_F)$$

$$\boxed{\approx 2\pi \Gamma^2 N_L(E_F)\,N_R(E_F) \int d\nu \; (n_F(\nu-V) - n_F(\nu))}$$

§ Equal time Green's function and the shape of Fermi surface

$$G(0^+,x) - G(-0^+,x) = \langle \{ c(x), c^\dagger(0)\}\rangle = \delta(x).$$

Let's pick up a direction $\hat{x}$, and study the large-x behavior of

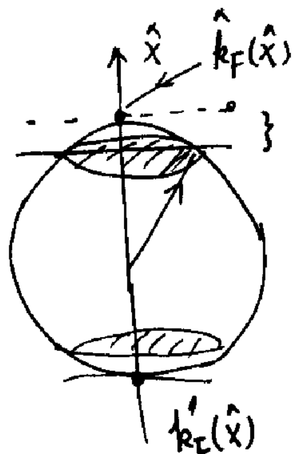
$$G(\pm 0^+, x\hat{x}).$$

At $T=0$, $G(-0^+, x) = -\int \frac{d^dk}{(2\pi)^d}\, n_F(\xi_k)\, e^{i\vec{k}\vec{x}}$

$$= \underbrace{-\int_{-\infty}^{+\infty} dk\; \tilde{N}(k,\hat{x})\, e^{ik|x|}}_{\text{integral along the } \hat{x}\text{-direction}}, \quad \text{where } \underbrace{\tilde{N}(k,\hat{x}) = \int \frac{d^dk}{(2\pi)^d}\,\Theta(-\xi_k)\,\delta(k - \vec{k}\cdot\hat{x})}_{\text{integral over the other "d-1" dimension}}$$

$\vec{k}\cdot\hat{x} = \text{const}$ defines a $\overbrace{\text{hyper-plane}}^{}$ (d-1 dim), which is tangential to Fermi surface at two points $k_F(\hat{x})$ and $k_F'(\hat{x})$

$\hat{N}(k,\hat{x})$ basically is cross-section area, intersected by the plane $\vec{k}\cdot\hat{x} = k$ and the Fermi sea. Without loss of generality, let us set $\hat{x}$ along the $\hat{z}$-direction



$\} \; \hat{k}_F(\hat{x})\cdot\hat{x} - k \rightarrow$ this a second order small number compare to the radius of the cross section

i.e. area $\propto \left(\sqrt{\left|\hat{k}_F(\hat{x})\cdot\hat{x} - k\right|}\right)^{d-1}$

Similar reason also applies to $k'_F(\hat{x}) \Rightarrow \hat{N}(k,\hat{x})$ has two singularity at two ends.

$$\Rightarrow \hat{N}(k,\hat{x}) = C_1 \, \Theta(k_F(\hat{x})\cdot\hat{x} - k)\,|\hat{k}_F(\hat{x})\cdot\hat{x} - k|^{\frac{d-1}{2}} \quad \text{(singluar parts)}$$

$$+ C_2\, \Theta(k'_F(\hat{x})\cdot\hat{x} - k)\,|-\hat{k}'_F(\hat{x})\cdot\hat{x} + k|^{\frac{d-1}{2}}$$

$$\Rightarrow iG(-0^+, x)\Big|_{x\to+\infty} = \int_{-\infty}^{+\infty} dk \; C_1\, \Theta(k_F(\hat{x})\cdot\hat{x} - k)\,|\hat{k}_F(\hat{x})\cdot\hat{x} - k|^{\frac{d-1}{2}} \, e^{ik|\hat{x}|} + \cdots$$

$$\propto \frac{C_1 e^{i\hat{k}_F(\hat{x})\cdot\hat{x}}}{|x|^{\frac{d+1}{2}}}\, e^{-i\frac{\pi(d+1)}{4}} + \frac{C_2\, e^{-ik_F(\hat{x})\cdot\hat{x}}}{|x|^{\frac{d+1}{2}}}\, e^{i\frac{\pi(d+1)}{4}}$$

(by introducing convegence factor $e^{ik|x|} \rightarrow e^{i(k+i0^+)|x|}$