path integral for dissipative system

Consider a QM system $H_p = \frac{p^2}{2m} + V(q)$. which furthur couples to an environment. The environment is described by a bounch of harmonic oscilattors $H_B[q_\alpha] = \sum_{\alpha=1}^{N} \left(\frac{\hat{P}_\alpha^2}{2m_\alpha} + \frac{m_\alpha}{2} \omega_\alpha^2 q_\alpha^2 \right)$, and the coupling between them is linear to q_α. as $H_c = - \sum_{\alpha=1}^{N} f_\alpha(q) q_\alpha$.

Then consider the survival probability of a particle confined to a metastable minimum at $q=a$

$$\langle a | e^{-i\hat{H}t} | a \rangle = \int_{q(0)=q(t)=a} Dq \; e^{i S_p[q]} \int Dq_\alpha \, e^{i S_B[q_\alpha] + i S_c[q, q_\alpha]}$$

where $S_p[q] = \int_0^t dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$, $S_B = \int_0^t dt \sum_\alpha \frac{m_\alpha}{2} \left(\dot{q}_\alpha^2 - \omega_\alpha^2 q_\alpha^2 \right)$

$$S_c[q] = - \int_0^t dt \sum_\alpha f_\alpha(q) q_\alpha$$

Now we change to Eucledian Rep, and integrate out environment degrees of freedom

$$\int Dq_\alpha \, e^{-\int_0^\beta d\tau \left[\left(\frac{m}{2} \frac{dq_\alpha}{d\tau} \right)^2 + \omega_\alpha^2 q_\alpha^2 + f_\alpha(q) q_\alpha \right]}$$

Peform Fourier transform $q(\tau) = \sum_{i\omega_n} q(i\omega_n) e^{-i\omega_n \tau}$

$$f_\alpha(q) = \sum_{i\omega_n} f_{\alpha, \omega_n}(q) \, e^{-i\omega_n \tau}$$

②

Then, the above Eq $= \int Dq_\alpha \, e^{-\beta \sum_{\omega_n} \left[\frac{m}{2}(\omega_n^2+\omega_\alpha^2) q_\alpha(i\omega_n) q_\alpha(-i\omega_n) + f_{\alpha,-\omega_n}(q)\, q_\alpha(i\omega_n) \right]}$

$$= \int Dq_\alpha \exp\left[-\beta \sum_{i\omega_n} \frac{m(\omega_n^2+\omega_\alpha^2)}{2} \left[q_\alpha(i\omega_n) + \frac{f_{\alpha,\omega_n}}{m(\omega_n^2+\omega_\alpha^2)} \right]\left[q_\alpha(-i\omega_n) + \frac{f_{\alpha,-\omega_n}}{m(\omega_n^2+\omega_\alpha^2)} \right]\right]$$

$$\cdot \exp\left[\beta \sum_{\omega_n} \frac{f_{\alpha,\omega_n}(q) f_{\alpha,-\omega_n}(q)}{2 m_\alpha (\omega_n^2+\omega_\alpha^2)} \right]$$

The integral over q_α is still Gaussian, which gives a q-independent result. Let us consider a simple case, that f_α is a linear function

$$f_\alpha[q(\tau)] = C_\alpha q(\tau) \longrightarrow f_{\alpha,\omega_n} = C_\alpha q(i\omega_n)$$

$$\Rightarrow S_{eff}[q] = S_p(q) - \frac{\beta}{2} \sum_{i\omega_n,\alpha} \frac{C_\alpha^2 q(i\omega_n) q(-i\omega_n)}{m_\alpha(\omega_n^2+\omega_\alpha^2)}$$

we would like to subtract a frequency independent term

$$\frac{1}{\omega_n^2+\omega_\alpha^2} \rightarrow \frac{1}{\omega_n^2+\omega_\alpha^2} - \frac{1}{\omega_\alpha^2} = \frac{-\omega_n^2}{\omega_\alpha^2(\omega_n^2+\omega_\alpha^2)}$$

The coupling shift the potential

$$V(q) \rightarrow V(q) + \left(\frac{1}{2}\sum_\alpha \frac{C_\alpha^2}{\omega_\alpha^2 m_\alpha}\right) q^2 ,$$

C carries the unit of $m\omega^2$.

This shift is non-interesting, we neglect it. then we ariveat

$$\boxed{\begin{aligned} &S_{eff}[q] = S_p[q] + \beta \sum_{i\omega_n} q(i\omega_n) K(i\omega_n) q(-i\omega_n), \text{ where} \\ &K(i\omega_n) = \frac{1}{2} \sum_\alpha \frac{C_\alpha^2 \omega_n^2}{m_\alpha \omega_\alpha^2 (\omega_\alpha^2+\omega_n^2)} \end{aligned}}$$

Define spectra density $J(\omega) = \frac{\pi}{2} \sum_\alpha \frac{C_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha)$

then $K(i\omega_n) = \int_0^{+\infty} \frac{d\omega}{\pi} J(\omega) \frac{\omega_n^2}{\omega(\omega^2 + \omega_n^2)}$. Consider a weight of

Ohmic damping $J(\omega) = \eta \omega$. $\boxed{J\text{'s unite} = [m\omega^2][\eta] = m\omega}$.

$$\boxed{K(i\omega_n) = \int_0^{+\infty} \frac{d\omega}{\pi} \eta \frac{\omega_n^2}{\omega^2 + \omega_n^2} = \frac{|\omega_n|\eta}{\pi} \int_0^\infty dx \frac{1}{x^2+1} = \frac{\eta}{2}|\omega_n|},$$

then

go back to time-domain

$$q(i\omega_n) = \frac{1}{\beta} \int_0^\beta d\tau\, q(\tau) e^{i\omega_n \tau}, \quad K(i\omega_n) = \frac{1}{\beta} \int_0^\beta d\tau'\, K(\tau') e^{-i\omega_n \tau'}$$

$$\rightarrow \beta \beta^{-3} \int_0^\beta d\tau \int_0^\beta d\tau' \int_0^\beta d\tau''\, q(\tau) q(\tau') \sum_{\omega_n} e^{i\omega_n(\tau - \tau' - \tau'')} K(\tau'')$$

$$\boxed{(*)\ \frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n(\tau-\tau'-\tau'')} = \sum_m \delta(\tau - \tau' - \tau'' + m\beta)}$$

$$= \beta^{-1} \int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau - \tau') q(\tau')$$

$$\omega_n = \frac{2n\pi}{\beta}$$

(~~we neglect~~ ($K(\tau)$ is also a periodical function of τ

$\tau - \tau'$ may go outside $[0, \beta]$, which is equivalent

to put $\tau - \tau'$ inside $[0, \beta]$, but keep all possible $m \neq 0$ in $(*)$]

$$\Rightarrow \boxed{S_{eff}[q] = S_p[q] + \beta^{-1} \int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau - \tau') q(\tau')}$$

K's unit: $[K] = [m\omega^2]$.

Now we calculate $K(\tau-\tau')$:

$$K(\tau) = \sum_n K(\omega_n) e^{i\omega_n \tau} = \frac{\eta}{2}\sum_n |\omega_n| e^{i\omega_n\tau} = \eta \sum_{n=1}^{\infty} \frac{2n\pi}{\beta} e^{in\frac{2\pi}{\beta}(\tau+i0^+)}$$

set $x = \frac{2\pi}{\beta}$, $\sum_{n=1}^{\infty} nx\, e^{inx(\tau+0^+)} = -i\frac{\partial}{\partial\tau}\sum_{n=1}^{\infty} e^{inx(\tau+i0^+)}$

$$= -i\frac{\partial}{\partial\tau}\sum_{n=0}^{\infty} e^{inx(\tau+i0^+)} = -i\frac{\partial}{\partial\tau}\frac{1}{1-e^{ix(\tau+0^+i)}}$$

$$= -i\frac{x(-)(-)ie^{ix\tau}}{(1-e^{ix\tau})^2} = \frac{x}{(e^{-i\frac{x\tau}{2}} - e^{\frac{i}{2}x\tau})^2} = \frac{x}{-4\sin^2\frac{x\tau}{2}}$$

$$\Rightarrow K(\tau) = \eta\frac{2\pi}{\beta}\frac{1}{-4\sin^2\frac{\pi}{\beta}(\tau+i0^+)} = -\frac{\pi}{2}\frac{\eta}{\beta}\frac{1}{\sin^2\frac{\pi(\tau+i0^+)}{\beta}}$$

at $\tau << \beta \Rightarrow K(\tau) = \frac{-\eta\beta}{2\pi(\tau+i0^+)^2}$

Further $q(\tau)K(\tau-\tau')q(\tau') = \left\{q^2(\tau) + q^2(\tau') - \frac{1}{2}[q(\tau)-q(\tau')]^2\right\}K(\tau-\tau')$

The first two terms again after integration only renormalize to $V(q)$, and will be droped. Then we arrive at

$$\boxed{S_{eff}[q] = \int_0^\beta d\tau\, \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q) + \frac{\eta}{4\pi}\int_0^\beta d\tau \int_0^\beta d\tau' \frac{(q(\tau)-q(\tau'))^2}{(\tau-\tau'+i0^+)^2}}$$

I

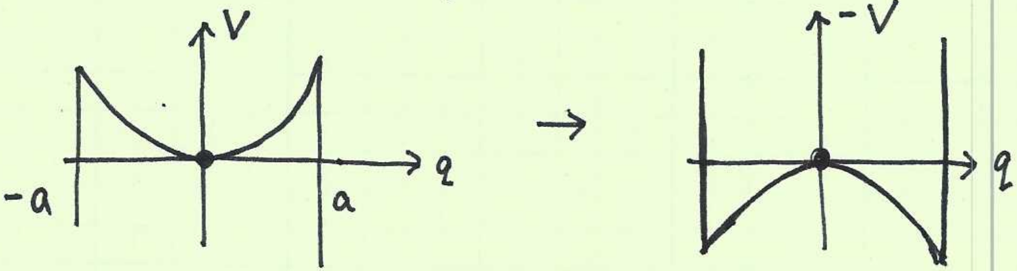
change back to Minkowski space $\tau \to it$

$$i S_{eff}[q(t)] = -S_{eff}(\tau \to it) = i\int dt \left[\frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q)\right] - \int dt\, dt' \,\frac{\eta}{4\pi} \frac{(q(t)-q(t'))^2}{(t-t')^2}$$

damping

$$\boxed{S[q(t)] = \int dt\, \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q) + i\int dt\, dt' \,\frac{\eta}{4\pi} \frac{(q(t)-q(t'))^2}{(t-t')^2}}$$

§2 Now let us come back to the survival probability



Consider the bounce solution $q(0) = q(\beta) \to 0$, at $\tau = \beta/2$. the particle bounce back. The classic equation of motion

$$-m\ddot{q} + m\omega_c^2 q + \frac{\eta}{\pi}\int_0^\beta d\tau' \,\frac{q(\tau)-q(\tau')}{(\tau-\tau')^2} = A\,\delta(\tau - \beta/2)$$

A is the coefficent to represent the discontinuous change of the velocity

LHS: $\sum_n m(\omega_n^2 + \omega_c^2)\, q(\omega_n)\, e^{-i\omega_n \tau} - \frac{2}{\beta}\int_0^\beta d\tau' \left[\sum_{n''} K(i\omega_{n''})\, e^{i\omega_{n''}(\tau-\tau')}\right] \cdot \left\{\sum_n q(\omega_n)\left(e^{-i\omega_n \tau} - e^{-i\omega_n \tau'}\right)\right\}$

$$-2\int_0^\beta d\tau' \frac{1}{\beta}\sum_{nn''} K(i\omega_{n''})\, q(\omega_n)\,(-)\,e^{i\omega_{n''}(\tau-\tau') - i\omega_n \tau'} = \eta \sum_n |\omega_n|\, q(\omega_n)\, e^{-i\omega_n \tau}$$

$$\int_0^\beta d\tau' \sum_{nn''} q(\omega_n)\, e^{-i\omega_n \tau}\, |\omega_{n''}|\, e^{-i\omega_{n''}(\tau-\tau')} = 0$$

$$\Rightarrow \text{LHS} = \sum_n \left\{ m(\omega_n^2 + \omega_c^2) + \eta|\omega_n| \right\} q(\omega_n)\, e^{-i\omega_n \tau}$$

$$\text{RHS} = \frac{1}{\beta} \sum_n A\, e^{-i\omega_n(\tau - \beta/2)}$$

$$\boxed{\left(m(\omega_n^2 + \omega_c^2) + \eta|\omega_n|\right) q(\omega_n) = \frac{A}{\beta}\, e^{i\omega_n \frac{\beta}{2}}}$$

Solution of classic equation.

$$\sum_n q(\omega_n)\, e^{-i\omega_n \frac{\beta}{2}} = \frac{A}{\beta} \sum_n \frac{1}{m(\omega_n^2 + \omega_c^2) + \eta|\omega_n|}$$

$$a = q\left(\frac{\beta}{2}\right) = Af, \quad \text{where } f \triangleq \frac{1}{\beta} \sum_n \frac{1}{m(\omega_n^2 + \omega_c^2) + \eta|\omega_n|}$$

boundry condition of bouncing action.

$$\text{or } A = \frac{a}{f}, \qquad [f] = \left[\frac{1}{m\omega^2}\right]$$

The action of the bouncing solutions

$$S_0 = \beta \sum_n \frac{m}{2} |q(\omega_n)|^2 (\omega_n^2 + \omega_c^2) + \beta^{-1} \int_0^\beta d\tau \int_0^\beta d\tau' \sum_n q(\omega_n)\, e^{-i\omega_n \tau} \sum_{n'} K(\omega_{n'})\, e^{i\omega_{n'}(\tau-\tau')} \sum_{n''} q(\omega_{n''})\, e^{-i\omega_{n''}\tau'}$$

$$= \beta \sum_n \frac{m}{2} |q(\omega_n)|^2 (\omega_n^2 + \omega_c^2)$$

$$+ \beta \sum |q(\omega_n)|^2 K(\omega_n) \leftarrow \frac{\eta}{2}|\omega_n|$$

$$\Rightarrow S_{\text{bounce}} = \frac{\beta}{2} \sum_n \left[m(\omega_n^2 + \omega_c^2) + \eta|\omega_n| \right] |q_n|^2$$

plug in $q(\omega_n) = \frac{A}{\beta}\, e^{i\omega_n \frac{\beta}{2}} \Big/ \left(m(\omega_n^2+\omega_c^2) + \eta|\omega_n|\right)$

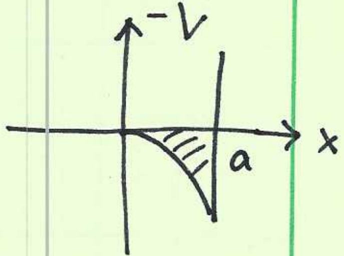
$$\Rightarrow S_{bounce} = \frac{A^2}{2\beta}\sum_n \frac{1}{(\ \cdots\ )} = \frac{A^2}{2} f = \boxed{\frac{a^2}{2f} = S_{bounce}}.$$

⊛ and the tunneling rate $\boxed{\Gamma \sim e^{-S_{bounce}}}$.

① Consider the limit $\eta \to 0$ and $\beta \to \infty$: zero temperature, no dissipation

$$f = \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{1}{m(\omega^2+\omega_c^2)} = \frac{1}{2\pi m\omega_c} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+1}$$

$$= \frac{1}{2m\omega_c} \Rightarrow S_{bounce} = m\omega_c a^2$$



$$S = 2\int_0^a dx \left(\frac{dx}{d\tau}\right) m = 2\int_0^a dx\ \omega_c x\, m = m\omega_c a^2$$

bound back and forth

so $\Gamma \sim e^{-m\omega_c a^2/\hbar}$ is controlled by the barrier height $\frac{m\omega_c^2 a^2}{2}$ at the attempt frequecy ω_c.

The classic solution

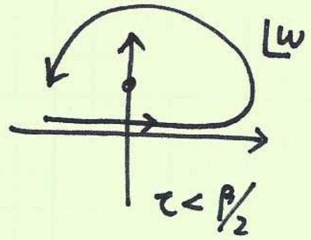
$$q(\tau) = \sum_{\omega_n} q(\omega_n) e^{-i\omega_n \tau} = \beta\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{A}{\beta} \frac{e^{-i\omega_n(\tau-\beta/2)}}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|} \leftarrow \eta=0$$

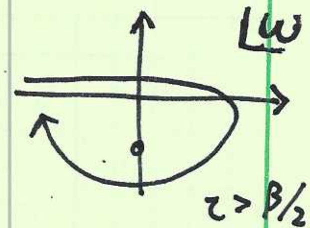
$$= \frac{a}{mf}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(\tau-\beta/2)}}{\omega^2+\omega_c^2} = \frac{-2\pi i}{2\pi}\frac{a}{mf} \lim\left(\frac{1}{\omega-\omega_c i}\right) e^{-i\omega(\tau-\beta/2)}\Bigg|_{\omega\to-\omega_c i} \quad \boxed{\text{at } \tau > \beta/2}$$

$$q(\tau) = -\frac{2\pi i}{2\pi}\,\frac{a}{mf}\,\frac{1}{-2\omega_c i}\,e^{-\omega_c|\tau-\beta/2|}$$ ← combine results at $\tau<\beta/2$ from residual theory

$$= \frac{a}{2\cancel{\pi} m f \omega_c}\, e^{-\omega_c|\tau-\beta/2|}$$

$$= a\, e^{-\omega_c|\tau-\beta/2|}$$





② consider $\beta \to +\infty$, and overdamped reigon $\eta >> m\omega_c$

$$f = \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|} = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{m(\omega^2+\omega_c^2)+\eta|\omega|}$$

in the regin :

the contribution ~~comes from $\frac{1}{\eta|\omega|}$~~

$$\left.\begin{array}{l}\eta|\omega| >> m\omega^2\\ \eta|\omega| \geq m\omega_c^2\end{array}\right\} \Rightarrow \frac{m\omega_c^2}{\eta} << |\omega| << \eta/m$$

$$\Rightarrow f_1 \simeq \frac{2}{2\pi\eta}\int_{m\omega_c^2/\eta}^{\eta/m} d\omega\,\frac{1}{\omega} = \frac{1}{\pi\eta}\ln\frac{\eta/m}{m\omega_c^2/\eta} = \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

At $|\omega| >> \eta/m$, we have $f' = \frac{1}{\pi m}\int_{\eta/m}^{+\infty} d\omega\,\frac{1}{\omega^2} = \frac{1}{\pi m}\,\frac{m}{\eta} \simeq \frac{1}{\pi\eta}$

$|\omega| << \frac{m\omega_c^2}{\eta} \quad\rightarrow\quad f'' = \frac{1}{\pi m}\int_0^{\frac{m\omega_c^2}{\eta}} \frac{1}{\omega_c^2} \simeq \frac{1}{\pi\eta}$

combine together, the contribution mainly from $\frac{m\omega_c^2}{\eta} << |\omega| << \eta/m$

$$\Rightarrow f \simeq \frac{2}{\pi\eta}\left[\ln\frac{\eta}{m\omega_c} + 1\right] \simeq \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

then $\boxed{S_{bounce} = \dfrac{\pi \eta a^2}{4 \ln \eta / m\omega_c}}$, at $\eta \gg m\omega_c$, this soultion

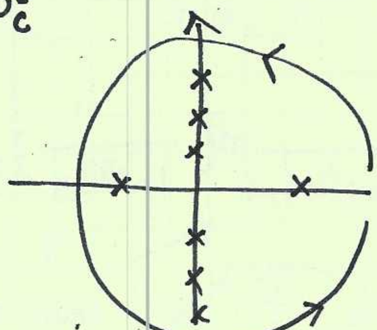
is much larger than the coherent case $S_{bounce} = m\omega_c a^2$.

$\Rightarrow$ tunneling probability is exponentially suppressed.

$\longrightarrow$ The particle becames classic, - no tunnely.

③ $\eta \to 0$ but $\beta \neq 0$, thermal : $f = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2 + \omega_c^2}$

define $\oint \frac{dz}{2\pi i} \frac{1}{e^{\beta z} - 1} \frac{1}{m(-\omega^2 + \omega_c^2)}$



$$0 = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2 + \omega_c^2} + \sum_i \mathrm{Res} \frac{1}{e^{\beta z} - 1} \frac{1}{m(-\omega^2 + \omega_c^2)} \Bigg|_{z = \pm \omega_c}$$

$$\Rightarrow f = +\left[\frac{1}{e^{\beta\omega_c} - 1} \frac{1}{2\omega_c} + \frac{1}{e^{-\beta\omega_c} - 1} \left(-\frac{1}{2\omega_c} \right) \right] = \frac{1}{2\omega_c m} \frac{e^{\beta\omega_c} + 1}{e^{\beta\omega_c} - 1} = \frac{\coth \frac{\beta\omega_c}{2}}{2m\omega_c}$$

$$\Rightarrow S = \frac{a^2}{\cancel{4}\left(\frac{\coth \frac{\beta\omega_c}{2}}{m\omega_c} \right)} = m\omega_c a^2 \cdot \tanh \frac{\beta\omega_c}{2} \to \begin{cases} m\omega_c a^2 \text{ at } \beta \to \infty \\ \frac{\beta}{2} m\omega_c^2 a^2 \text{ at } \beta \to 0 \end{cases}$$

at $\beta \to 0$ (high T): $P \sim e^{\frac{m\omega_c^2 a^2}{2 k_B T}}$ (classic thermal activated behavior!)