Lect 2: Path integral — instantons (warm up).

§1. Consider a single particle potential $H = \frac{p^2}{2m} + V(x)$.

We use the imaginary time path integral

$$\boxed{\begin{aligned} \langle x_f | e^{-HT/\hbar} | x_i \rangle &= A \int [Dx(t)]\, e^{-\frac{S}{\hbar}} \\ &= A \int Dx(t) \exp\left\{ - \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left[\frac{m}{2} \left(\frac{dx}{d\tau}\right)^2 + V(x) \right] \right\}. \end{aligned}}$$

A_N: Const from integral measure.

Search for classic paths with the following boundary condition

$$x\left(\tfrac{T}{2}\right) = x_f \quad \text{and} \quad x\left(-\tfrac{T}{2}\right) = x_i$$

$\Rightarrow$ classic path $\boxed{\bar{x}(\tau): \quad \frac{\delta S_c}{\delta \bar{x}} = -m \frac{d^2 \bar{x}}{d\tau^2} + \frac{d}{dx} V(x) \Big|_{x = \bar{x}(t)} = 0}$

Consider small fluctuations. $x(\tau) = \bar{x}(\tau) + \delta\bar{x}(\tau)$

with $\delta \bar{x}(\tau) = 0$ at $\tau = -T/2$, and $T/2$

$$S = \int \left[\frac{m}{2} \left(\frac{dx}{d\tau}\right)^2 + V(x) \right] d\tau = \int \left(\frac{m}{2} \frac{d\bar{x}}{d\tau} + V(\bar{x}) \right) d\tau + \int d\tau \, \frac{m}{2} \left(\frac{d\,\delta\bar{x}}{d\tau} \right)^2 + \frac{1}{2} V''(\bar{x}(\tau)) (\delta \bar{x})^2$$

where $V'' = \frac{d^2 V}{dx^2}$

$$= S_c(\bar{x}(t)) + \int_{-T/2}^{T/2} d\tau \, \frac{1}{2} \, \delta x(\tau) \left[-m \frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right] \delta x(\tau)$$

$$\Rightarrow \langle x_f | e^{-HT/\hbar} | x_i \rangle \simeq A\, e^{-\frac{S(\bar{x}(\tau))}{\hbar}} \cdot \left\{ \det\left[-m\frac{d^2}{d\tau^2} + V''(\bar{x}(t)) \right] \right\}^{-1/2}$$

Suppose we can solve the eigen spectra and orth-normal eigenfunction

$$\left\{ -m\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right\} x_n(\tau) = \lambda_n x_n(\tau) \text{ with } x_n(-\tfrac{T}{2}) = x_n(\tfrac{T}{2}) = 0$$

then we express $\delta x(\tau) = \sum_n C_n\, x_n(\tau)$

$$\Rightarrow \int_{-T/2}^{T/2} d\tau\, \frac{1}{2}\, \delta x(\tau) \left[-m\frac{d^2}{d\tau^2} + V'' \right] \delta x_n(\tau) = \frac{1}{2}\sum_n \lambda_n C_n^2$$

$$\rightarrow \int Dx(t)\, e^{-\delta S/\hbar} \simeq \prod_n \int_{-\infty}^{+\infty} dC_n\, e^{-\frac{1}{2}\lambda_n C_n^2} \simeq \boxed{\prod_n \lambda_n^{-1/2} = \left\{ \det\left[-m\frac{d^2}{d\tau^2} + V'' \right] \right\}^{-1/2}}$$

where we drop all the ~~anstants~~ constants.

Example: consider a harmonic potential $V(x) = \frac{1}{2} m\omega^2 x^2$, and $x_i = x_f = 0$. Calculate $\boxed{\langle 0 | e^{-HT/\hbar} | 0 \rangle = \left[\frac{m\omega}{2\pi\hbar \sinh \omega T} \right]^{1/2}}$.

It's obvious that $\bar{x}(\tau) = 0$. For fluctuations

$$\left\{ -m\frac{d^2}{d\tau^2} + m\omega^2 \right\} x_n(\tau) = \lambda_n x_n(\tau)$$

$$\Rightarrow x_n(\tau) = \begin{cases} \cos\left(\frac{n\pi}{T}\tau\right)\frac{1}{\sqrt{2T}}, & n = 1, 3, \cdots \\ \sin\left(\frac{n\pi}{T}\tau\right)\frac{1}{\sqrt{2T}}, & n = 2, 4, \cdots \end{cases} \Rightarrow \lambda_n = m\left[\omega^2 + \left(\frac{n\pi}{T}\right)^2 \right]$$

$$\Rightarrow \langle 0| e^{-HT/\hbar} |0\rangle = A \prod_{n=1}^{\infty} \left[m \left(\omega^2 + \left(\frac{n\pi}{T}\right)^2\right)\right]^{-1/2} = A' \prod_{n=1}^{\infty} \left[1 + \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}$$

we donot know A' but it can be calibrated through the free space result:

$$\underbrace{\langle 0| e^{-iHt/\hbar} |0\rangle}_{\text{Set } \omega \to 0} = \left(\frac{m}{2\pi i t\hbar}\right)^{1/2} \longrightarrow \left(\frac{m}{2\pi T\hbar}\right)^{1/2} = A'_N$$

$$\Rightarrow \langle 0| e^{-HT/\hbar} |0\rangle = \left(\frac{m}{2\pi T\hbar}\right)^{1/2} \left[\prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega T}{n\pi}\right)^2\right)\right]^{-1/2} = \left(\frac{m}{2\pi T\hbar}\right)^{1/2} \left(\frac{\sinh \omega T}{\omega T}\right)^{-1/2}$$

$$= \left[\frac{m\omega}{2\pi\hbar \sinh \omega T}\right]^{1/2}.$$

$$\boxed{\text{Lemma to calculate } \det\left[-m\frac{d^2}{d\tau^2} + V''\right]:}$$

Suppose $\left[-m\frac{d^2}{d\tau^2} + W(t)\right]\psi = \lambda\psi$, where ψ satisfies the boundary condition

$$\begin{cases} \psi(-T/2) = 0 \\ \frac{d}{d\tau}\psi\Big|_{\tau = -T/2} = 1 \end{cases}.$$

Let $W_1(t)$ and $W_2(t)$ be two functions of t, and $\psi_{1,2}(t)$ be the corresponding solutions for the above boundary condition; then we have

$$\boxed{\frac{\det[-m\partial_\tau^2 + W_1(t) - \lambda]}{\det[-m\partial_\tau^2 + W_2(t) - \lambda]} = \frac{\psi_{1,\lambda}(\frac{T}{2})}{\psi_{2,\lambda}(\frac{T}{2})}}.$$

Proof: ~~treat~~ λ as a continous variable. Both sides are semi-analytical functions of λ. As $\lambda \to \lambda_{1n}$, $\psi_{1,\lambda_{1n}}(\frac{T}{2}) \to 0$, and $\lambda \to \lambda_{2,n}$, $\psi_{2,\lambda_{2,n}}(\frac{T}{2}) \to 0$ as assumed.

In these cases, $\det[-m\partial_\tau^2 + W_{1,2}(t) - \lambda] \longrightarrow 0$ for $\lambda \to \lambda_{1n}$ and λ_{2n}, respectively. Then both sides have the same pattern of zeros and poles, thus they must be equal up to a constant. Then as $|\lambda| \to \infty$, both sides $\to 1$, $\Rightarrow$ LHS = RHS.

Exercise: use this method, calculate $\langle 0| e^{-HT/\hbar} |0\rangle$ for $V = \frac{1}{2} m\omega^2 x^2$ again.

$$\langle 0| e^{-HT/\hbar} |0\rangle = A \left[\det\left[-m\frac{d^2}{d\tau^2} + m\omega^2\right]\right]^{-1/2}.$$

$$\left[\frac{\det \frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}{\det \frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2}\right)}\right] = \frac{\psi_{\omega,\lambda=0}(T/2)}{\psi_{\omega=0,\lambda=0}(T/2)}.$$

For $\omega = 0$, $\lambda = 0$ $\Rightarrow$ $\psi_{\omega=0,\lambda=0}(\tau) = \tau + T/2$

ω, $\lambda = 0$ $\Rightarrow$ $\left(\frac{d^2}{d\tau^2} - \omega^2\right)\psi(\tau) = 0$ $\Rightarrow$ $\psi_{\omega,\lambda=0}(\tau) = \frac{\sinh(\omega(\tau + T/2))}{\omega}$

$$\Rightarrow \frac{\det m\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}{\det m\left(-\frac{d^2}{d\tau^2}\right)} = \left.\frac{\sinh\omega(\tau + T/2)}{\omega(\tau + T/2)}\right|_{\tau = T/2} = \frac{\sinh\omega T}{\omega T}$$

For $\det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right]$:

$$\langle 0| e^{-HT/\hbar} |0\rangle_{\omega=0} = A \det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right]$$

$$= \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \cdot \langle 0|p\rangle\langle p|0\rangle \, e^{-p^2T/2m\hbar} = \frac{1}{2\pi\hbar}\left(\frac{\pi}{T/2m\hbar}\right)^{1/2}.$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2}$$

$$\Rightarrow \langle 0| e^{-HT/\hbar} |0\rangle = A \det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right] \frac{\det\left[\frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\right]}{\det\left[\frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2}\right)\right]}$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2}\left(\frac{\sinh\omega T}{\omega T}\right)^{-1/2} = \left[\frac{m\omega}{2\pi\hbar \sinh\omega T}\right]^{1/2}$$

$$\xrightarrow[T\to\infty]{} \frac{m\omega}{\pi\hbar}\, e^{-\omega T/2}.$$

Compare with $\langle 0| e^{-HT/\hbar} |0\rangle \longrightarrow |\langle 0|G\rangle|^2 e^{-E_0 T/\hbar} = \left\{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\right\}^2 e^{-\omega T/2}$.