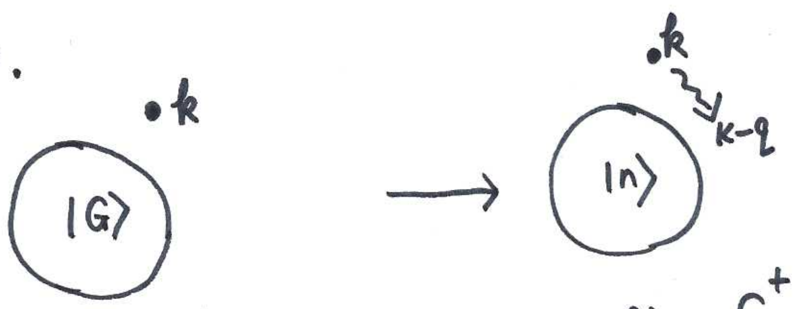
# Interacting electron gas – life time, Fermi surface

## § Fermi golden rule — many body version

Consider put an electron in the plane wave state $\vec{k}$ outsider the ground state $|G\rangle$ (interacting systems). This is not the eigenstate, and the $\vec{k}$-electron will decay, say, to $\vec{k}-\vec{q}$, and the Fermi sea is also excited to be $|n\rangle$.




$$|ini\rangle = c_k^{+}|G\rangle \qquad\qquad |f\rangle = c_{k-q}^{+}|\;n\rangle$$

The scattering Hamiltonian $H'_{int} = \frac{1}{V} v(q)\, c^{+}_{k-q} c_k \rho^{+}_q$, where $k, k-q > k_F$. The frequency transfer $\omega_q = \frac{1}{\hbar}(\varepsilon_k - \varepsilon_{k-q})$.

The Fermi golden rule (transition rule)

$$W(\vec{q}, \omega_q) = \frac{2\pi}{\hbar} \sum_n |\langle f| H'_{int} |i\rangle|^2 \,\delta[\hbar\omega_q - (E_f - E_i)]$$

— Check unit, correct!

Summing over $\vec{q}$, we have

$$\frac{1}{\tau_k} = \sum_q{}' W(\vec{q}, \omega_q) = \frac{2\pi}{\hbar V^2} \sum_q v^2(q) \sum_n \langle n| \rho_q^{+} |0\rangle\, \delta(\hbar\omega_q - (E_n - E_0))$$

where $|n\rangle = |f\rangle$, and we'll use $\hbar\omega_{n0} = E_n - E_0$. $\sum_q{}'$ means the constraint on phase space with $\varepsilon_{k-q} > 0$ and $\varepsilon_k - \varepsilon_{k-q} > 0$.

Remember the sum rule

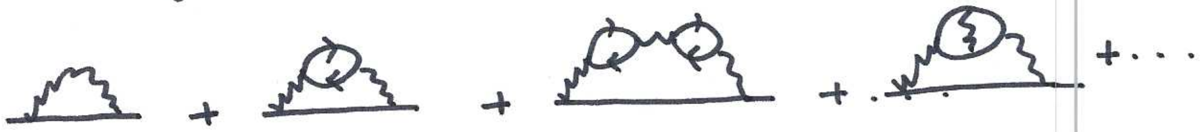
$$\mathrm{Im}\,\frac{1}{\varepsilon(q,\omega)} = \frac{\pi v(q)}{\hbar V}\sum_n |\langle n|\rho_q|0\rangle|^2\big(\delta(\omega+\omega_{n0})-\delta(\omega-\omega_{n0})\big).$$

for isotropic system, $|\langle n|\rho_q|0\rangle|^2 = |\langle n|\rho_{-q}|0\rangle|^2 = |\langle n|\rho_q^+|0\rangle|^2$

$$\Rightarrow\ W(\vec{q},\omega_q) = \frac{-2v(q)}{V\hbar}\ \mathrm{Im}\,\frac{1}{\varepsilon(q,\omega_q)} \quad \text{for } \omega_q = \varepsilon_k-\varepsilon_{k-q} > 0.$$

$$\boxed{\frac{1}{\tau_k} = \frac{-2}{\hbar V}\sum_q{}' v(q)\ \mathrm{Im}\,\frac{1}{\varepsilon(q,\omega_q)}\ ,\ \text{with } \omega_q = \varepsilon_k-\varepsilon_{k-q}}$$

§ From Green's function point of view $\frac{1}{\tau_k} = -2\,\mathrm{Im}\,\Sigma_{ret}(k,\varepsilon_k)$. Let's derive it digrammatically.



$$\Sigma(k,ik_n) = -\frac{1}{V}\sum_q v_q \frac{1}{\beta}\sum_{iq_n}\frac{\mathcal{G}^0(k+q,ik_n+iq_n)}{\varepsilon(q,iq_n)}$$

use spectra representation

$$\mathcal{G}^0(k+q,ik_n+iq_n) = \int_{-\infty}^{+\infty}\frac{d\varepsilon'}{2\pi}\,\frac{A(k+q,\varepsilon')}{ik_n+iq_n-\varepsilon'}$$

$$\frac{v_q}{\varepsilon(q,iq_n)} = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,\frac{B(q,\omega')}{iq_n-\omega'}$$

and $-\frac{1}{\beta}\sum_{iq_n}\frac{1}{ik_n+iq_n-\varepsilon'}\,\frac{1}{iq_n-\omega'} = \frac{n_B(\omega')+n_F(\varepsilon')}{ik_n+\omega'-\varepsilon}$ please check!

$$\Rightarrow \Sigma(k, ik_n) = \frac{1}{V}\sum_{q}\int_{-\infty}^{+\infty}\frac{d\varepsilon'}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\, A(k+q,\varepsilon')\, B(q,\omega')\, \frac{n_B(\omega') + n_F(\varepsilon')}{ik_n + \omega' - \varepsilon'}$$

at $T=0$, $n_B(\omega') = -\Theta(-\omega')$, & $n_F(\epsilon') = \Theta(-\epsilon')$

$$\Sigma_{ret}(k,\varepsilon) = \int\frac{d\vec{q}}{(2\pi)^3}\left[\int_{-\infty}^{0}\frac{d\epsilon'}{2\pi}\int_{0}^{+\infty}\frac{d\omega'}{2\pi} - \int_{0}^{+\infty}\frac{d\epsilon'}{2\pi}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\right]\frac{A(k+q,\epsilon')\, B(q,\omega')}{\varepsilon+\omega'-\varepsilon'+i\eta}$$

The spectra functions: $A(k+q,\varepsilon') = 2\pi\delta(\varepsilon' - \varepsilon_{k+q})$

$$B(q,\omega') = -2\,\mathrm{Im}\,\frac{v_q}{\varepsilon_{ret}(q,\omega'+i\eta)}$$

$$\Rightarrow \Sigma_{ret}(k,\varepsilon) = \int\frac{d\vec{q}}{(2\pi)^3}\left[\int_{-\infty}^{0}\frac{d\epsilon'}{2\pi}\int_{0}^{+\infty}\frac{d\omega'}{2\pi} - \int_{0}^{+\infty}\frac{d\epsilon'}{2\pi}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\right] 2\pi\delta(\epsilon'-\epsilon_{k+q})\,(-2)\,\mathrm{Im}\,\frac{v_q}{\varepsilon_{ret}(q,\omega'+i\eta)}$$

$$\times\frac{1}{\varepsilon+i\eta+\omega'-\varepsilon'}$$

For bosonic freqnency spectra function, it's odd respect to $\omega'$

$$\Rightarrow \Sigma_{ret}(k,\varepsilon) = -\int\frac{d^3\vec{q}}{(2\pi)^3}\int_{0}^{+\infty}\frac{d\omega'}{\pi}\left\{\Theta(-\epsilon_{k+q})\frac{1}{\varepsilon+i\eta+\omega'-\varepsilon_{k+q}} + \Theta(\varepsilon_{k+q})\frac{1}{\varepsilon+i\eta-\omega'-\varepsilon(k+q)}\right\}$$

$$v_q\,\mathrm{Im}\,\frac{1}{\varepsilon_{ret}(q,\omega')}$$

$$\mathrm{Im}\,\Sigma_{ret}(k,\varepsilon) = \begin{cases}\displaystyle\int\frac{d^3\vec{q}}{(2\pi)^3}\int_{0}^{+\infty}d\omega'\,\Theta(\varepsilon_{k+q})\,\delta(\varepsilon-\omega'-\varepsilon(k+q))\, v_q\,\mathrm{Im}\,\frac{1}{\varepsilon_{ret}(q\omega')} & \text{for } (\varepsilon>0). \\ \displaystyle\int\frac{d^3\vec{q}}{(2\pi)^3}\int_{0}^{+\infty}d\omega'\,\Theta(-\varepsilon_{k+q})\,\delta(\varepsilon+\omega'-\varepsilon_{k+q})\, v_q\,\mathrm{Im}\,\frac{1}{\varepsilon_{ret}(q\omega')} & (\text{for } \varepsilon<0)\end{cases}$$

$$\mathrm{Im}\,\Sigma_{ret}(k,\varepsilon) = \begin{cases} \displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(\varepsilon_{k+q})\, \Theta(\varepsilon - \varepsilon_{k+q})\, v_q\, \mathrm{Im}\left(\frac{1}{\varepsilon_{ret}(q\ \ )}\right) & \varepsilon > 0 \\ \displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(-\varepsilon_{k+q})\, \Theta(\varepsilon_{k+q} - \varepsilon)\, v_q\, \mathrm{Im}\, \frac{1}{\varepsilon_{ret}(q, \varepsilon - \varepsilon_{k-q})} & \varepsilon < 0 \end{cases}$$

$\varepsilon - \varepsilon_{k-q}$

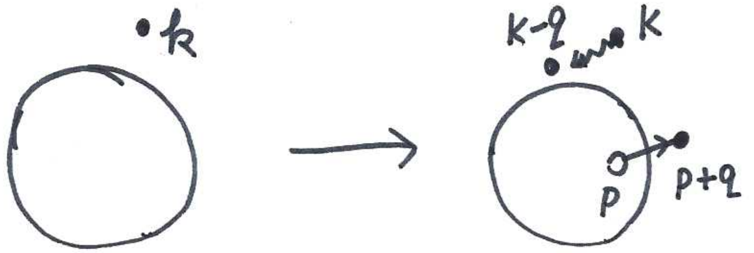
take $k > k_f$, $\varepsilon = \varepsilon_k > 0$, $\Rightarrow$ (also change $q \to -q$)

$$\mathrm{Im}\,\Sigma_{ret}(k, \varepsilon_k) = \underbrace{\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(\varepsilon_k - \varepsilon_{k-q})\, \Theta(\varepsilon_{k-q})}_{\frac{1}{V}\sum'_q}\, v_q\, \mathrm{Im}\, \frac{1}{\varepsilon_{ret}(q, \varepsilon_k - \varepsilon_{k-q})}$$

$$\Rightarrow \boxed{\frac{1}{\tau_k} = -2\,\mathrm{Im}\,\Sigma_{ret}(k, \varepsilon_k)}$$

3: Change a view — screened Conlumb potential.

In this picture, we view the Fermi sea as free electron plane wave Slater determinaint state. The states after scattering are also plane-waves, but the interaction we will use is the screened Conlumb.




This is equivalent to change a representation, but the matrix elements are the same.

Fermi golden rule $\displaystyle \frac{1}{\tau_k} = \sum_q{}' W(\vec{q}, \varepsilon_k - \varepsilon_{k-q})$

$$= \frac{2\pi}{\hbar^2 V} \sum_q{}' \sum_{p\sigma} \frac{v^2(q)}{|\varepsilon(q, \varepsilon_k - \varepsilon_{k-q})|^2}\, n_p (1 - n_{p+q})\, \delta(\omega - (\varepsilon_k - \varepsilon_{k-q}))$$

Remember in the Lindhardt function:

$$\chi^{o}_{ret}(q,\omega) = -\frac{2}{V}\sum_{k}\frac{n_k - n_{k+q}}{\hbar(\omega-\omega_{kq})+i\eta} \quad \leftarrow n_k - n_{k+q} = n_k(1-n_{k+q}) - n_{k+q}(1-n_k)$$

The positive $\omega$ part $\rightarrow$ $Im\,\chi^{o}_{ret}(q,\omega) = \frac{2}{\hbar V}\sum_{K}\pi n_K(1-n_{K+q})\,\delta(\omega-\omega_{kq})$

For the RPA form $\varepsilon(q,\omega) = 1+\frac{4\pi e^2}{q^2}\chi^{o}(q,\omega+i\eta)$

$$\Rightarrow Im\,\varepsilon(q,\omega) = \frac{\pi}{\hbar V}v(q)\sum_{k\sigma} n_k(1-n_{k+q})\,\delta(\omega-\omega_{kq}) \quad \text{for } \omega>0.$$

$$\Rightarrow \frac{1}{\tau_K} = \frac{2}{\hbar V}\sum_q{}' v(q)\frac{Im\,\varepsilon(q,\omega_q)}{|\varepsilon(q,\omega_q)|^2} = \boxed{-\frac{2}{\hbar V}\sum_q{}' v(q)\, Im\frac{1}{\varepsilon_{RPA}(q,\omega_q)} = \frac{1}{\tau_K} \qquad \omega_q = \omega_K - \omega_{K+q}}$$

§4: Now we ready to do real calculation

$$Im\,\varepsilon(q,\omega\to 0) = \frac{\pi}{2}\left(\frac{k_{TF}}{q}\right)^2\frac{\omega}{v_F q} \rightarrow \frac{\pi}{2}\left(\frac{k_{TF}}{q}\right)^2\frac{(\varepsilon_K-\varepsilon_{K-q})}{\hbar v_F q}$$

for small frequency, we can have $Re\,\varepsilon(q,\omega) \gg Im\,\varepsilon(q,\omega)$, and we can set $Re\,\varepsilon(q,\omega) \simeq \varepsilon(q,0)$. $\Rightarrow$

$$\frac{1}{\tau_K} = \frac{\pi}{V\hbar}\sum_q{}' v(q)\left(\frac{k_{TF}}{q}\right)^2\frac{(\varepsilon_K-\varepsilon_{K-q})}{\hbar q v_F|\varepsilon(q,0)|^2}$$

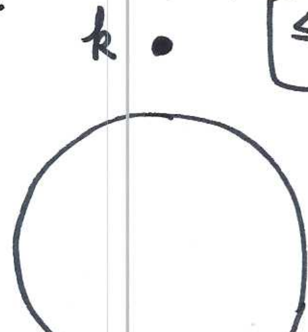
in the long wave length limit, $\varepsilon(q,0) = 1+\frac{k_{TF}^2}{q^2}$

and $\omega_K - \omega_{K-q} \simeq \frac{k_F q}{m}\cos\theta$, where $\theta$ is the angle between $\hat{k}$ and $\hat{q}$.

we also need the constraint

$$\varepsilon_k > \varepsilon_{k-q} > \varepsilon_F$$

$\leftarrow k^2+q^2-2kq\cos\theta > k_f^2 \Rightarrow \boxed{\cos\theta \le \frac{k^2-k_f^2}{2kq}}$




$$\Rightarrow \quad 0 \le \cos\theta \le \frac{k^2-k_F^2}{2k_f q} \equiv z_m$$

$$\Rightarrow \frac{1}{\tau_k} = e^2 k_{TF}^2 \int_0^{2k_f} dq \int_0^{z_m} d\cos\theta \, \frac{\cos\theta}{q^2\left(1+\frac{k_{TF}^2}{q^2}\right)^2}$$

$$= \frac{e^2 k_{TF}^2}{8k_f^2}\,(k^2-k_f^2)^2 \int_0^{2k_f} dq \, \frac{1}{(q^2+k_{TF}^2)^2}$$

$\leftarrow$ the upper limit of $q = 2k_f$ should not be considered seriously. The TF approximation for $\varepsilon(q,0)$ only valid at $q \ll k_f$.

$$= \frac{e^2 k_{TF}^2}{8k_f^2}\,(k^2-k_f^2)^2\, k_{TF}^{-3} \int_0^{+\infty} dx \, \frac{1}{(x^2+1)^2}$$

$$\approx \frac{e^2}{8k_{TF}} \, \frac{4k_f^2\,(k-k_f)^2}{k_f^2} \cdot \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{1}{\tau_k} = \frac{\pi}{8}\,\frac{e^2}{k_{TF}}\,\frac{\varepsilon_f^2}{v_F^2}\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2 = \frac{\pi}{32}\,\frac{e^2 k_f^2}{k_{TF}}\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2$$

$$\sim \frac{\pi^2\sqrt{3}}{128}\,\omega_p\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2,$$

where $\omega_p^2 = \frac{4\pi\rho e^2}{m}$ is the plasma frequency.

$$\Rightarrow \operatorname{Im}\Sigma_{ret}(k,\varepsilon) = -\frac{\pi^2\sqrt{3}}{256}\,\omega_p\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2$$

— long life time well-defined quasi-particle.

Lect 9

# Landau Fermi liquid theory (I)

## § Concept of quasi-particles

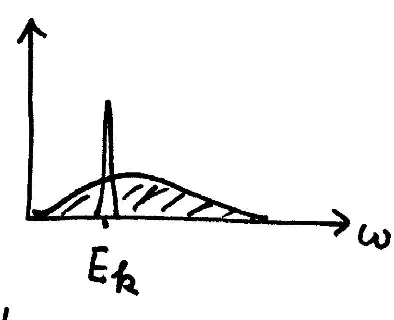
Suppose we have a many-body fermion ground state $|G\rangle$. At time $t=0$, an extra particle is added at the plane wave state $C_k^\dagger$. After a time interval of $T$, we check what's the amplitude remaining in this state.

$$G_k(t) = \langle G| e^{iHT} C_k \, e^{-iHT} C_k^\dagger | G \rangle = \langle G | C_k(T) \, C_k^\dagger(0) | G \rangle$$

in terms of Lehmann's representation

$$G_k(t) = \sum_m \langle G | C_k(T) | m \rangle \langle m | C_k^\dagger(0) | G \rangle = \sum_m |\langle G | C_k | m \rangle|^2 e^{-i(E_m - E_g)T}$$

In the Fermi liquid state, the distribution of the spetral weight has a special value of $\delta$-peak (may be brodened), and a continuum (incoherent part)




$$Z = |\langle \psi_k | C_k | G \rangle|^2 \leftarrow$$ wavefunction renormalization factor

$$\Rightarrow G_k(t) = Z \, e^{-iE_k' T} \leftarrow$$ quasi-particle

$$+ \int \frac{d\omega}{2\pi} A(\omega) \, e^{-i\omega t}$$

"$0 < Z < 1$" justifies the validity of Fermi liquid state. Even though in an interacting system, we can still look it as if a free system. Quasi-particle is like "a running horse running in a dusty road, dressed by a cloud of dust

§ Physical content of the self-energy $P_{249}$ Negele & Orland.

no-interacting Green function $G_0(k,\omega) = \dfrac{1}{\omega - \varepsilon_k + i\,\mathrm{sgn}(\omega)\eta}$ → single pole with residue "1".

For the full green's function

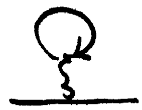
$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma(k,\omega)},$$

we need to exam the structures of poles and the residues.

Let us only exam the first two order pertubation theory

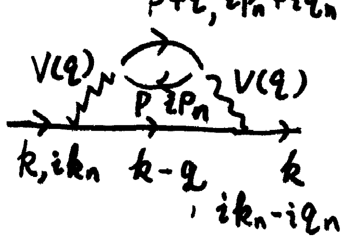
$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma_1(k) - \Sigma_2(k,\omega)}$$

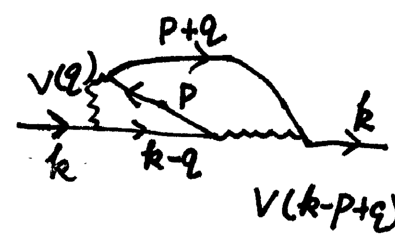
The first order is HF which is frequency independent

$\Sigma_1(k) =$  $+$  $= \sum_{k'} v(q=0)\, n_{k'} - V(k-k')\, n_{k'}$.

( $v$ has no fregency dependence → $\Sigma_1(k)$ has no frequency dependence.

The second order

$\Sigma_2(k,\omega) =$ 

$+$ 


frequency summation

$$\frac{1}{\beta}\sum_{iq_n}\left[\frac{1}{\beta}\sum_{ip_n}\frac{1}{ip_n+iq_n-\varepsilon_{p+q}}\;\frac{1}{ip_n-\varepsilon_p}\right]\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

$$= \frac{1}{\beta}\sum_{iq_n}\frac{n_f(\varepsilon_p)-n_f(\varepsilon_{p+q})}{iq_n-(\varepsilon_{p+q}-\varepsilon_p)}\;\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

define $$S = \frac{1}{\beta}\sum_{iq_n} \frac{1}{iq_n - (\mathcal{E}_{p+q} - \mathcal{E}_p)} \frac{1}{ik_n - iq_n - \mathcal{E}_{k-q}}$$

& $$f(z) = \frac{1}{z - (\mathcal{E}_{p+q} - \mathcal{E}_p)} \frac{1}{ik_n - z - \mathcal{E}_{k-q}}$$

$$I = \lim_{R\to\infty} \int \frac{dz}{2\pi i} f(z) \frac{1}{e^{\beta z}+1} = 0. \Rightarrow$$

$$\frac{1}{\beta}\sum_n f(iq_n) + \frac{1}{e^{\beta(\mathcal{E}_{p+q}-\mathcal{E}_p)}-1} \frac{1}{ik_n - (\mathcal{E}_{p+q} - \mathcal{E}_p + \mathcal{E}_{k-q})}$$

$$+ \frac{1}{e^{\beta(ik_n - \mathcal{E}_{k-q})}-1} \frac{-1}{ik_n - (\mathcal{E}_{p+q} - \mathcal{E}_p + \mathcal{E}_{k-q})} = 0$$

$$\Rightarrow \frac{1}{\beta}\sum_n f(iq_n) = -\frac{[n_B(\mathcal{E}_{p+q}-\mathcal{E}_p) + 1 - n_f(\mathcal{E}_{k-q})]}{ik_n - (\mathcal{E}_{p+q} - \mathcal{E}_p + \mathcal{E}_{k-q})}$$

$$\Rightarrow \Sigma_2(k,\omega) = \sum_p \sum_q \frac{[n_f(\mathcal{E}_p) - n_f(\mathcal{E}_{p+q})][1 - n_f(\mathcal{E}_{k-q}) + n_B(\mathcal{E}_{p+q}-\mathcal{E}_p)]}{\omega + i\eta - (\mathcal{E}_{p+q} - \mathcal{E}_p + \mathcal{E}_{k-q})} [V^2(q) - V(q)V(k-p+q)]$$

real frequency

$\omega > 0$ for $k > k_f$.

$\Sigma_2(k,\omega)$ explicitly depends on $\omega$. $\Sigma_2$ has an infinite number of poles at $\omega = \mathcal{E}_{p+q} - \mathcal{E}_p + \mathcal{E}_{k-q}$ $\longrightarrow$ finite imaginary part.

☆ Let us consider a simple example. If $\Sigma_2(k,\omega)$ has two poles

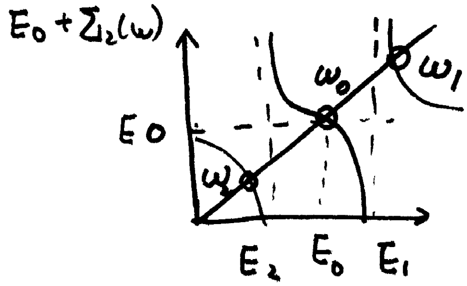
$$\Sigma_2(\omega) = \frac{A_1}{\omega - E_1 + i\eta} + \frac{A_2}{\omega - E_2 + i\eta},$$ then what's the poles and residues of $$G(\omega) = \frac{1}{\omega - E_0 - \Sigma_2(\omega)}$$

④

we assume $E_1$ is above $E_0$, and $E_2$ is below $E_0$, and the residues in $\Sigma_2$ are very small, satisfying $\frac{A_1}{|E_1-E_0||E_2-E_0|}, \frac{A_2}{|E_1-E_0||E_2-E_0|} \ll 1$.

Since $\Sigma_2$ is small at $\omega \neq E_1, E_2$, we first consider $\Sigma_1$, which gives the order of the pole of $\omega = E_0$. Including $\Sigma_2$, we solve $\omega = E_0 + \Sigma_2(\omega)$

$\omega_0$ is close to $E_0$; $\omega_{1,2}$ close to $E_{1,2}$.




We expand $G(\omega)$ around each poles

$$\omega_i = E_0 + \Sigma_2(\omega_i) \qquad i = 0, 1, 2$$

$$\omega - E_0 - \Sigma_2(\omega) = \omega - E_0 - \Sigma_2(\omega_i) + (\omega - \omega_i)\Sigma_2'(\omega_i)$$

$$= (\omega - \omega_i)(1 - \Sigma_2'(\omega_i)) \Rightarrow G(\omega) \simeq \frac{1}{1-\Sigma_2'(\omega)} \frac{1}{\omega - \omega_i + i\eta}$$

$$\frac{1}{1-\Sigma_2'(\omega_0)} = \frac{1}{1+\sum_i \frac{A_i}{(\omega_0 - E_i)^2}} \approx 1 - \sum_{i=1}^{2} \frac{A_i}{(E_0-E_i)^2}$$

$$\frac{1}{1-\Sigma_2'(\omega_1)} = \frac{1}{1 + \frac{A_1}{(\omega_1 - E_1)^2} + \frac{A_2}{(\omega_1 - E_2)^2}} \approx \frac{(\omega_1 - E_1)^2}{A_1}$$

consider $\omega_1 = E_0 + \frac{A_1}{\omega_1 - E_1} + \frac{A_2}{\omega_1 - E_2}$

$$\Rightarrow \omega_1 - E_0 \approx \frac{A_1}{\omega_1 - E_1} \quad \text{or} \quad \omega_1 - E_1 \approx \frac{A_1}{\omega_1 - E_0}$$

$$\Rightarrow \frac{1}{1-\Sigma_2'(\omega_1)} \approx \frac{A_1}{(E_1-E_0)^2}, \quad \text{similarly} \quad \frac{1}{1-\Sigma_2'(\omega_2)} \approx \frac{A_2}{(E_2-E_0)^2}.$$

⑤

After switching on interaction, the single pole is fragmented into three poles. The major one is still close to $E_0$, but with a smaller residue $1 - \sum_i \frac{A_i}{(E_0 - E_i)^2}$. This pole is called the quasi-particle pole.

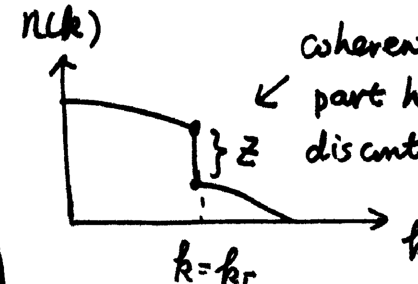
The ~~strength~~ removed from the quasi-particle pole has been redistributed to poles at $\omega_{1,2}$. These poles represent complicated many-body medium excitations of

Now consider the real case.

$$\boxed{\left.\frac{1}{1 - \frac{\partial}{\partial\omega}\Sigma_2(\omega)}\right|_{\omega \simeq E_0} \approx 1 - \sum_{p,q} \frac{A_{p,q}^2}{[E_0 - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{p-q})]^2} = Z}$$

depleted to the incoherent background.

The Landau fermi-liquid is based on the assuption that $Z$ remains finite. $\rightarrow$




coherent part has discontinuity

Now let's consider damping

$$\frac{1}{\tau} = -2\,\mathrm{Im}\,\Sigma_2 = 2\pi \sum_{p,q} \underbrace{|A_{p,q}|^2}\, \delta[\omega - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})]$$

$$C_k^\dagger C_k = \underbrace{Z}_{\text{WF renormalization}} \underbrace{\tilde{C}_k^\dagger \tilde{C}_k}_{\text{quasi-particle}} + \cdots$$

$$\frac{1}{\tau} \approx |V|^2 \underbrace{\int_0^{\varepsilon_k} d\epsilon_{k-q}}_{\text{particle}} \underbrace{\int_0^{\epsilon_k - \epsilon_{k-q}} d\epsilon_{p+q}}_{\text{particle}} \rho(\epsilon_{k-q})\,\rho(\epsilon_{p+q})\,\rho(\underbrace{\varepsilon_p = -\varepsilon_k + (\varepsilon_{p+q} + \varepsilon_{k-q})}_{\text{hole}})$$

$$\leq |V|^2 \rho_{max}^3\, \varepsilon_k^2 \quad \Longrightarrow \quad \boxed{\tau(\epsilon) \propto |\epsilon - \epsilon_F|^{-2}}$$

☆ effective mass.

$$\mathcal{E} = \frac{k^2}{2m} - \mu + \Sigma(k, \mathcal{E})$$

$$\frac{d\epsilon}{dk} = \frac{k}{m} + \frac{\partial \Sigma}{\partial k} + \frac{\partial \Sigma}{\partial \mathcal{E}} \frac{d\mathcal{E}}{dk} \Rightarrow \frac{d\mathcal{E}}{dk} = \left(1 - \frac{\partial \Sigma}{\partial \epsilon}\right)^{-1} \left(\frac{k}{m} + \frac{\partial \Sigma}{\partial k}\right)$$

$$= \left(1 - \frac{\partial \Sigma}{\partial \epsilon}\right)^{-1} \frac{k}{m} \left(1 + \frac{m}{k} \frac{\partial \Sigma}{\partial k}\right)$$

define $\frac{d\epsilon}{dk} = \frac{k}{m^*}$ $\Rightarrow$ $\boxed{m^* = m\left(1 + \frac{m}{k}\frac{\partial \Sigma}{\partial k}\right)^{-1} \times \left(1 - \frac{\partial \Sigma}{\partial \omega}\right)\Big|_{\omega = \mathcal{E}}}$