Lect 12: two-body correlation functions & linear responses

§1 density-density correlation

$$i\,P^{00}(t,x) = \langle T\,(\rho(t,x)\,\rho(0,0))\rangle \qquad \rho(x,t) = c^{\dagger}(t,x)\,c(0,0)$$

by Wick theorem $\Rightarrow$ $i\,P^{00}(t,x) = \langle T\,[\,c^{\dagger}(t+0^{+},x)\,c(t,x)\,c^{\dagger}(0^{+})\,c(0)\,]\rangle$

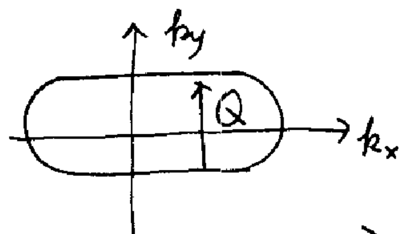
$$= \rho_0^{2} - i\,G(-t,-x)\; i\,G(t,x)$$

as $t\to 0$, for spherical Fermi surface, we have

$$iG(-0^{+},x)\big|_{x\to+\infty} \sim \cos\!\left(k_F|x| - \frac{\pi(d+1)}{4}\right)\frac{1}{|x|^{(d+1)/2}}$$

$$\Rightarrow\; i\,P^{00}(t,x) - \rho_0^{2} \;\longrightarrow\; \left[1 - \sin\!\left(2k_F|x| - \frac{\pi(d+1)}{2}\right)\right]\frac{1}{|x|^{d+1}}\,.$$

For nested Fermi surface



the effective dimension is reduced to $d=1$

$$\Rightarrow\; \left(1 + \sin(Q|x|)\right)\frac{1}{|x|^{2}}$$

$\hookrightarrow$ algebraic long range crystal.

Retarded density-density response:

$$i\,\Pi^{00}(t,q) = \frac{\theta(t)}{V}\left\langle\left[\,\sum_{k}\left(c^{\dagger}_{k}\,c_{q+k}\right)(t),\; \sum_{k'} c^{\dagger}_{k'}\,c_{k'-q}(0)\right]\right\rangle$$

$$= \frac{\Theta(t)}{V} \sum_{k} (1-n_F(\xi_{q+k}))\, n_F(\xi_k)\, e^{-it(\xi_{q+k}-\xi_k)}$$

$$- (1-n_F(\xi_k))\, n_F(\xi_{q+k})\, e^{+it(\xi_k-\xi_{q+k})}$$

$$\rightarrow \Pi^{00}(\omega, q) = V^{-1} \sum_{k} \frac{-(n_F(\xi_{q+k}) - n_F(\xi_k))}{\omega - (\xi_{q+k}-\xi_k) + i0^+}$$

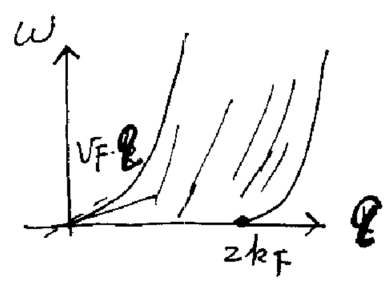
as $|q| \ll k_F$

$$\Pi^{00}(\omega, q) = +\int \frac{d^d q}{(2\pi)^d} \frac{\partial n_F}{\partial \xi} \frac{-\vec{v}\cdot\vec{q}}{\omega - \vec{v}\cdot\vec{q} + i0^+} \; ; \; \left(\frac{\partial n_F}{\partial \xi} = -\delta(\xi_k) \text{ at } 0\,K \right).$$

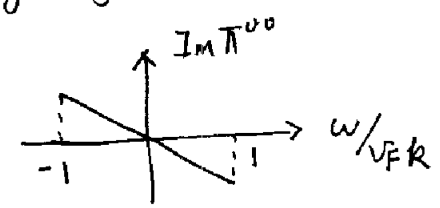
$$\mathrm{Im}\,\Pi^{00}(\omega, q) = -s N_0 \int \frac{d\Omega}{4\pi} \delta(s - \cos\theta) \quad (\text{for } 3d)$$ (N_0 is the density of states).

where $s = \frac{\omega}{v_F q}$, N_0 is the density of states.

$$= -\frac{\pi}{2} N_0 s\, \Theta(-|s|+1).$$



Imaginary part means dissipation:



the figure of particle hole continuum.

how about Real part, at 3d, we have

$$\mathrm{Re}\,\Pi^{00}(\omega, q) = N_0 \int \frac{d\Omega}{4\pi} \frac{\cos\theta}{s - \cos\theta\ \text{[illegible]}}$$

$$= -N_0 \left[1 - \frac{s}{2} \ln\left| \frac{1+s}{1-s} \right| \right].$$

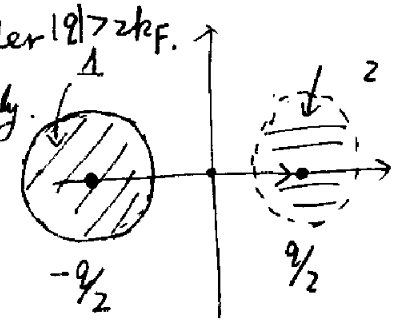
The compressbility $\chi = -\lim_{q\to 0} \Pi^{00}(q, 0) = N_0$

~~optical conductivity~~ $\cancel{\sigma(\omega) = -\lim_{k\to 0} \mathrm{Im} \frac{\omega}{k^2} \Pi^{00}(\omega, k) = N_0 \frac{\omega^2}{k^2}}$

how about $\Pi^{00}(\omega, q)$ as $|q| \to 2k_F$

$$\Pi^{00}(\omega, q) = \int \frac{d^d k}{(2\pi)^d} \frac{n_F(\xi_{k-q/2}) - n_F(\xi_{k+q/2})}{\omega - (\xi_{k+q/2} - \xi_{k-q/2}) + i0^+}$$

Let us translate the Fermi sphere, and consider $|q| > 2k_F$. left and right hand side with $\mp \frac{\vec{q}}{2}$, respectively.



if k is inside the shaded area 1,

then $n_F(\xi_{k+q/2}) = 1$, $n_F(\xi_{k-q/2}) = 0$.

$(q > 2k_F)$

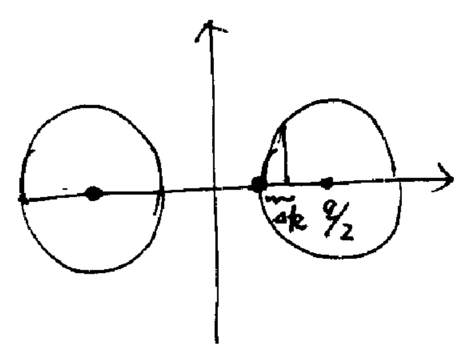
if k is inside the area 2 $\Rightarrow$

$n_F(\xi_{k-q/2}) = 1$, $n_F(\xi_{k+q/2}) = 0$.

If $\omega > 0$, we need $\xi_{k+q/2} > \xi_{k-q/2}$ to make the denominator vanish, The smallest value $\xi_{k+\frac{q}{2}} - \xi_{k-q/2} = \hbar^2(|q - k_F|^2 - k_F^2)/2m = \omega_0$

i.e. we take the region 2

as $\omega > \omega_0$, the area of the intersection is $\mathrm{Im}\,\Pi^{00}(\omega, q > 2k_F) \propto \pi\left(\sqrt{k_F \cdot \Delta k}\right)^2$



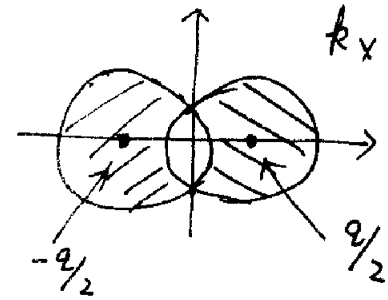
$$\Delta\omega = \left\{(q - k_F + \Delta k)^2 - (k_F - \Delta k)^2\right\} - (\omega_0)$$

$$= q \cdot \Delta k$$

$\Rightarrow$ at 3D, $\mathrm{Im}\,\Pi^{00}(\omega, q > k_F) \propto \Theta(\omega - \omega_0)\,|\omega|$, and for general dimensions d,

$$\propto \Theta(\omega - \omega_0)\,|\omega|^{\frac{d-1}{2}}.$$

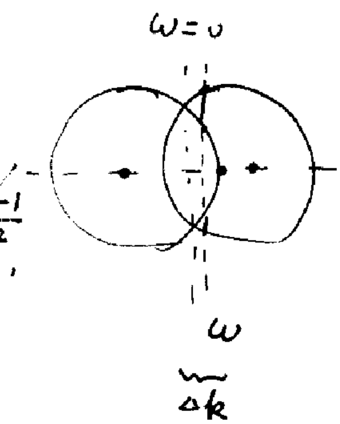
If $|q| < 2k_F$, after we translate the Fermi surface to $-\frac{q}{2}$ and $\frac{q}{2}$, respectively, the two new Fermi surfaces have a overlap region.



Let us calculate area satisfying $\omega = \xi_{k+q/2} - \xi_{k-q/2} > 0$

$\cancel{\text{when } \omega > \xi(k_F) - \xi(q-k_F) = \omega_0,}$

$\cancel{\text{we again have } \mathrm{Im}\,\Pi^0(q,\omega) \propto \Theta(\omega-\omega_0)\,(\omega-\omega_0)^{\frac{d-1}{2}},}$



as $0 < \omega < \omega_0$, $\Rightarrow \Delta k = \frac{\omega}{q}$

the area is a ring,

$\omega_0 = \xi(k_F) - \xi(q-k_F)$

the out radius: $\sqrt{k_F^2 - (\frac{q}{2} - \Delta k)^2}$

inner radium $\sqrt{k_F^2 - (\frac{q}{2} + \Delta k)^2}$

$$\Rightarrow \mathrm{Im}\,\Pi^{00}(\omega, |q| < 2k_F) \propto \left(\frac{q}{2} + \Delta k\right)^2 - \left(\frac{q}{2} - \Delta k\right)^2 = q \cdot \Delta k$$

$$\propto \omega .$$

as $\omega > \omega_0$, the cross-section is a circle, the area is

$\text{Area} = k_F^2 - (\frac{q}{2} - \Delta k)^2$, while $\omega \propto (\frac{q}{2} + \Delta k)^2 - (\frac{q}{2} - \Delta k)^2 \propto q \cdot \Delta k$

$\sim \left[k_F^2 - \left(\frac{q}{2}\right)^2\right] + q \cdot \Delta k \quad \Rightarrow \mathrm{Im}\,\Pi^{00}(\omega, |q| < 2k_F) \sim \left[k_F^2 - \left(\frac{q}{2}\right)^2\right] + \underline{\omega}$

§ Current - current correlation function

$$i\Pi^{ij}(t,x) = \Theta(t)\,\langle [j^i, j^j] \rangle \Rightarrow$$

$$\Pi^{ij}(\omega, q) = \int \frac{d^dk}{(2\pi)^d} \frac{[n_F(\xi_k) - n_F(\xi_{k+q})]\, v^i(k+q/2)\, v^j(k-q/2)}{\omega - (\xi_{k+q} - \xi_k) + i0^+}$$

At zero ~~temperature~~, and $q \to 0 \Rightarrow$ (at 3d)

$$\Pi^{ij}(\omega, q) = N_0 \int \frac{d\Omega}{4\pi} \frac{\hat{v}_F \cdot \vec{q}\;\; v_F^i v_F^j}{\omega - \hat{v}_F \cdot \vec{q} + i0^+}$$

$$= N_0 \widehat{v_F^2} \int \frac{d\Omega}{4\pi} \frac{\cos\theta \cdot \hat{\Omega}^i \hat{\Omega}^j}{s - \cos\theta + i0^+} \qquad (\text{we set } q \text{ along } \hat{z})$$

The longitudinal part can be infered from Π^{00}, or directly calculate $\Pi^{zz} = N_0 v_F^2 \int \frac{d\Omega}{4\pi} \frac{\cos^3\theta}{s - \cos\theta + i0^+}$.

The transverse part can be calculated as Π^{xx} or Π^{yy}

$$\Rightarrow \Pi^{xx} = \Pi^{yy} = \frac{N_0 v_F^2}{2} \int \frac{d\Omega}{4\pi} \frac{\cos\theta \sin^2\theta}{s - \cos\theta + i0^+}$$

$$\Rightarrow \text{longitudinal } \Pi^{zz}(\omega, q) = \begin{cases} -2s^2 - i\pi s^3 & (s \ll 1) \\ \frac{2}{3} + \frac{2}{5}\left(\frac{1}{s}\right)^2 & (s \gg 1) \end{cases}$$

$$\Pi^{xx}(\omega, q) = \Pi^{yy}(\omega, q) = \begin{cases} 2s^2 - i\frac{\pi}{2} s & (s \ll 1) \\ \frac{2}{3} + \frac{2}{15}\left(\frac{1}{s}\right)^2 & (s \gg 1) \end{cases}$$

§ Conductivity

We assume there's impurity scattering, which results in a finite lift time of particle.

$$i\,G(t,k) = \Theta(t)\,\Theta(\xi_k)\,e^{-it\xi_k - t\Gamma} - \Theta(-t)\,\Theta(-\xi_k)\,e^{-it\xi_k - (-t)\Gamma}$$

$$i\,G^\beta(t,k) = \Theta(t)\,(1-n_F(\xi_k))\,e^{-it\xi_k - t\Gamma} - \Theta(-t)\,n_F(\xi_k)\,e^{-it\xi_k - (-t)\Gamma}$$

$$\Rightarrow G^\beta(\omega,k) = G_+^\beta + G_-^\beta = \frac{1-n_F(\xi_k)}{\omega - \xi_k + i\Gamma} + \frac{n_F(\xi_k)}{\omega - \xi_k - i\Gamma}$$

$$\pi A_+(\omega,k) = \Gamma\,\frac{1-n_F(\xi_k)}{(\omega-\xi_k)^2 + \Gamma^2}, \quad \pi A_-(\omega,k) = \Gamma\,\frac{n_F(\xi_k)}{(\omega-\xi_k)^2+\Gamma^2}$$

however, this approximation doesn't satisfy

$$A_+^\beta(\omega) = e^{\beta\omega} A_-^\beta(\omega),$$

which means the above approx isn't self-consistent, but we can cure this by define.

$$\pi A_+(\omega,k) = \Gamma\,\frac{(1-n_F(\omega))}{(\omega-\xi_k)^2+\Gamma^2}, \quad \pi A_-(\omega,k) = \Gamma\,\frac{n_F(\omega)}{(\omega-\xi_k)^2+\Gamma^2}$$

Let us calculate $\Theta(t)\,\langle[\rho(t,q),\ \rho(0,-q)]\rangle$

$$= \langle [\,C_k^\dagger(t)\,C_{k+q}(t),\ C_{k+q}^\dagger(0)\,C_k(0)\,]\rangle$$

$$\cong \langle C_{k+q}(t)\,C_{k+q}^\dagger(0)\rangle \langle C_k^\dagger(t)\,C_k(0)\rangle - \langle C_{k+q}^\dagger(0)\,C_{k+q}(t)\rangle \langle C_k(0)\,C_k^\dagger(t)\rangle$$

$$= i\,G(t,k+q)\,(-i\,G(-t,k)) - (-i\,G(-t,k+q))^* \,(i\,G(t,k))^*$$

expand G in terms of spectra representation

$$G_+(\omega,k) = \int dt\,\Theta(t)\,G(t,k)\,e^{i\omega t} = \int d\nu\,\frac{A_+(\nu,k)}{\omega - \nu + i0^+}$$

~~3. non-linear sigma m.~~

$$G_-(\omega, k) = \int dt\, \Theta(-t)\, G(t,k)\, e^{i\omega t} = \int d\nu \frac{A_-(\nu, k)}{\omega - \nu + i0^+}$$

$$\Rightarrow \operatorname{Im} \Pi^{00}(\omega, q) = \frac{\pi}{V} \int d\nu \sum_k \left(A_-(\nu, k+q)\, A_+(\nu-\omega, k) - A_+(\nu, k+q)\, A_-(\nu-\omega, k) \right)$$

Similarly

$$\operatorname{Im} \Pi^{ij}(\omega, q) = \frac{\pi}{V} \int d\nu \sum_k v^i(k \quad) v^j(k \quad) \big(A_-(\nu, k+q/2)\, A_+(\nu-\omega, k-q/2) - A_+(\nu, k+q/2)\, A_-(\nu-\omega, k-q/2) \big)$$

$$\sigma^{ij}(\omega) = - \lim_{q \to 0} \frac{\Pi^{ij}(\omega, q)}{\omega} = -\int \frac{d\nu}{\pi} \int \frac{dk}{(2\pi)^d} \frac{\Gamma^2 v^i(k)\, v^j(k)\, (n_F(\nu) - n_F(\nu-\omega))}{\omega\, ((\nu - \xi_k)^2 + \Gamma^2)\, ((\nu - \omega - \xi_k)^2 + \Gamma^2)}$$

set $\omega \to 0$

$$\sigma^{ij}(\omega \to 0) = \int \frac{d\nu}{\pi} \int \frac{d^d k}{(2\pi)^d} \frac{\Gamma^2 v^i(k)\, v^j(k) \left(-\frac{\partial n_F(\nu)}{\partial \nu}\right)}{\left((\nu - \xi_k)^2 + \Gamma^2\right)^2}$$

$$\approx \int \frac{d^d k}{(2\pi)^2}\, \tau\, v^i(k)\, v^j(k)\, \delta(\xi_k) .$$

where $\tau = \dfrac{1}{2\Gamma}$