lecture 11. BCS theory and Bogoliubov method

§ BCS Hamiltonian

$$H = \sum_{k\sigma} \xi_{k\sigma} a^{\dagger}_{k\sigma} a_{k\sigma} - \frac{\overbrace{g(k_1 k_2; q)}}{2} \sum_{k_1 k_2 q} a^{\dagger}_{-k_2\,\sigma} a^{\dagger}_{-k_2+q\,\sigma'} a_{-k_1+q\,\sigma'} a_{k_1\sigma},$$ where we assume an attractive interaction

BCS ansatz:

$$\Psi(r_1 r_2 \cdots r_N; \sigma_1 \cdots \sigma_N) = N^{-1/2} A\{\varphi(r_1 r_2; \sigma_1\sigma_2)\,\varphi(r_3 r_4; \sigma_3\sigma_4) \cdots \varphi(r_{N-1} r_N; \sigma_{N-1}\sigma_N)\}$$

where A is antisymmetrization between the pairs $(r_1, \sigma_1 \rightleftarrows r_3, \sigma_3$, eg$)$ and $\varphi(r_1 r_2; \sigma_1\sigma_2)$ is antisymmetric for the exchange of $(r_1\sigma_1 \rightleftarrows r_2\sigma_2)$.

For singlet superconductor

$$\varphi(r_1 r_2; \sigma_1\sigma_2) = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)\,\varphi(|r_1 - r_2|)$$

where $\varphi(|r_1 - r_2|) = \sum_k \chi(\vec{k})\, e^{ik(r_1 - r_2)} \Rightarrow$

$$\varphi(r_1 r_2; \sigma_1\sigma_2) = \frac{1}{\sqrt{2}} \sum_k \chi(\vec{k})\,[(k\uparrow)_1(-k\downarrow)_2 - (-k\downarrow)_1(k\uparrow)_2]$$

i.e. $\chi(\vec{k}) = \chi(-\vec{k})$.

In the second quantization language.

$$\varphi(r_1 r_2; \sigma_1\sigma_2) = \sum_k \chi(\vec{k})\, a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow}|vac\rangle,$$ the many-body state can be written

$$\Psi_N = \underset{\substack{\uparrow \\ \text{normalization factor}}}{N^{-1/2}} \left[\sum_k \chi(\vec{k})\, a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow}\right]^{N/2} |vac\rangle,$$ such a wavefunction has a fixed number N.

BCS method is to release the fixed particle-number constraint $H-\mu N$.

$$\psi = \exp\left[\sum_k \chi(k)\, a^+_{k\uparrow} a^+_{-k\downarrow}\right] |vac\rangle = \prod_k \exp\left[\chi(k)\, a^+_{k\uparrow} a^+_{-k\downarrow}\right] |vac\rangle$$

$$= \left(\prod_k (1+\chi_k\, a^+_{k\uparrow} a^+_{-k\downarrow})\right)|vac\rangle$$

$\longrightarrow$ any higher order term vanishes due to $(a^+_{k\uparrow})^2 = (a^+_{-k\downarrow})^2 = 0$.

The fixed particle number constraint can be put by the chemical potential.

§: pairing states

Let's just consider the pair of state of $k\uparrow$, and $-k\downarrow$, they can support four states $|0_\uparrow 0_\downarrow\rangle$, $|0_\uparrow 1_\downarrow\rangle$, $|0_\downarrow 1_\uparrow\rangle$, $|1_\uparrow 1_\downarrow\rangle$.

the $\Psi = \prod_k \phi_k$, where $\phi_k = |0_\uparrow 0_\downarrow\rangle_k + \underbrace{\;}_{\chi_k} |1_\uparrow 1_\downarrow\rangle_k \xrightarrow[\text{normaliz}]{} u_k |0_\uparrow 0_\downarrow\rangle_k + v_k |1_\uparrow 1_\downarrow\rangle_k$

with $|u_k^2| + |v_k^2| = 1$. we can set u_k to be real and positive.

The parameter v_k can be determined up to a phase $v_k e^{i\varphi}$, where φ is independent of k. But for different pairs, φ has to be the same, which describes $\underbrace{\text{the phase}}_{\text{global}}$ coherence.

we can define

$$\Psi(\varphi) = \prod_k |u_k| |0_\uparrow 0_\downarrow\rangle + |v_k| e^{i\varphi} |1_\uparrow 1_\downarrow\rangle$$

then by doing $\Psi_N = \int_0^{2\pi} \Psi(\varphi) \exp\{-i \frac{N}{2} \varphi\}\, d\varphi$, where Ψ_N is the particle number eigenstate, $\Psi(\varphi)$ is the phase state.

§: optimization of the BCS Wave function

$$\langle \sum_{k\sigma} (\xi_k - \mu) a^+_{k\sigma} a_{k\sigma} \rangle + \langle H_{int} \rangle = \sum_{k\sigma} \epsilon_k \langle n_{k\sigma} \rangle + \langle H_{int} \rangle$$

$$\langle K \rangle = 2 \sum_k \epsilon_k |v_k|^2$$

$$\langle H_{int} \rangle = \frac{1}{2} \sum_{k_1 k_2} g(k_1 k_2) \langle a^+_{+k_2 \sigma} a^+_{-k_2 \sigma'} a_{-k_1 \sigma'} a_{k_1 \sigma} \rangle$$

$$\langle a^+_{k_2\sigma} a^+_{-k_2\sigma'} a_{-k_1\sigma'} a_{k_1\sigma} \rangle = \langle a^+_{k_2\sigma} a^+_{-k_2\bar\sigma} \rangle \langle a_{-k_1\bar\sigma} a_{k_1\sigma} \rangle .$$ (BCS Channel decoupling)

$$\Rightarrow \langle H_{int} \rangle = \sum_{k_1 k_2} g(k_1 k_2) \langle a^+_{k_2\uparrow} a^+_{-k_2\downarrow} \rangle \langle a_{-k_1\downarrow} a_{k_1\uparrow} \rangle$$

$$\langle 0_\uparrow 0_\downarrow | a_{-k_1\downarrow} a_{k_1\uparrow} | 1_\uparrow 1_\downarrow \rangle = u^*_{k_1} v_{k_1} = F_{k_1}$$

$$\Rightarrow \langle H_{int} \rangle = -\sum_{k_1 k_2} g(k_1 k_2) F_{k_1} F^*_{k_2} , \quad \text{where } F_{k_1} = u^*_{k_1} v_{k_1}$$

$$\Rightarrow \langle | H - \mu N | \rangle = 2 \sum_k \xi_k v_k^2 - \sum_{k_1 k_2} g(k_1, k_2) u^*_{k_1} v_{k_1} u_{k_2} v^*_{k_1}$$

using parametrization $u_K = \cos\theta_K$, $v_k = \sin\theta_K$ $\Rightarrow$

$$\langle | H - \mu N | \rangle = 2 \sum_k \xi_k \sin^2\theta_K - 1/4 \sum_{k_1 k_2} g(k_1 k_2) \sin 2\theta_{K_1} \sin 2\theta_{K_2}$$

$$= \sum_k \xi_k \left[\cos 2\theta_K + 1\right] - 1/4 \sum_{k_1} \sin 2\theta_{K_1} \sum_{K_2} g(k_1 k_2) \sin 2\theta_{K_2}$$

do variation

$$0 = \frac{\partial}{\partial\theta_k} \langle | H - \mu N | \rangle = +2\xi_k \sin 2\theta_k - \tfrac{1}{2}\cdot 2\cos 2\theta_k \sum_{k'} g(k\,k')\sin 2\theta_{k'} = 0$$

define $\Delta_k = \sum_{k'} g(k,k')\frac{1}{2}\sin 2\theta_{k'}$ $\Rightarrow$

$$\xi_k \sin 2\theta_k - \cos 2\theta_k \Delta_k = 0 \Rightarrow \tan 2\theta_k = \frac{\Delta_k}{\xi_k}$$

$$\cos 2\theta_k = \frac{\xi_k}{\sqrt{\Delta_k^2+\xi_k^2}}, \quad \sin 2\theta_k = \frac{\Delta_k}{\sqrt{\Delta_k^2+\xi_k^2}}$$

the gap equation (BCS)

$$\boxed{\Delta_k = \sum_{k'} g(k,k')\frac{\Delta_{k'}}{\sqrt{\Delta_k^2+\xi_k^2}}}, \quad \text{where } \xi_k = \varepsilon_k - \mu .$$

§ Bogoliubov - method

$$H_{MF} = \sum_{k\sigma} \xi_{k\sigma} a^+_{k\sigma} a_{k\sigma} + \sum_k [a^+_{k\uparrow} a^+_{-k\downarrow}\Delta_k + a_{-k\downarrow} a_{k\uparrow}\Delta^*_k]$$

$$+ \sum_{k_1 k_2} \frac{g(k_1 k_2)}{\cdot} \langle a^+_{k_1\sigma} a^+_{-k_1\bar\sigma}\rangle \langle a_{-k_2\bar\sigma} a_{k_2\sigma}\rangle$$

where $\Delta_k = \frac{-1}{V}\sum_{k'} g(k\,k')\, a_{-k'\downarrow} a_{k'\uparrow}$

Thus for each momentum k, $H = \sum_k (a^+_{k\uparrow}\; a_{-k\downarrow}) \begin{pmatrix} \xi_k & \Delta_k \\ \Delta^*_k & -\xi_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix}$

$$+ \xi_k + \sum_{kk'} \frac{g(kk')}{2} \langle a^+_{k\sigma} a^+_{-k\bar\sigma}\rangle \langle a_{-k'\bar\sigma} a_{k\sigma}\rangle$$

introduce Bogoliubov transformation

$$\begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix}$$

$$H = \sum_k (\alpha^+_{k\uparrow}\ \beta_{-k\downarrow}) \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} \xi_k & \Delta_k \\ \Delta^*_{-k} & -\xi_k \end{bmatrix} \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{bmatrix} + \cdots$$

$$= \sum_k (\alpha^+_{k\uparrow}\ \beta_{-k\downarrow}) \begin{bmatrix} \xi_k \cos 2\theta_k + \Delta_k \sin 2\theta_k, & -\xi_k \sin 2\theta_k + \Delta_k \cos 2\theta_k \\ -\xi_k \sin 2\theta_k + \Delta_k \cos 2\theta_k & -\xi_k \cos 2\theta_k - \Delta_k \sin 2\theta_k \end{bmatrix} \begin{bmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{bmatrix}$$

$\boxed{\text{Set } \Delta_k \text{ to be real}}$

set $\tan 2\theta_k = \dfrac{\Delta_k}{\xi_k}$ $\quad \cos 2\theta_k = \dfrac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}}$ $\quad \sin 2\theta_k = \dfrac{\Delta_k}{\sqrt{\xi_k^2 + \Delta_k^2}}$

$$\Rightarrow H = \sum_k (\alpha^+_{k\uparrow}\ \beta_{-k\downarrow}) \begin{pmatrix} E_k & \\ & -E_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{pmatrix} = \sum_k (\hbar\omega_k - 1/2)(\alpha^+_{k\uparrow}\alpha_{k\uparrow} + \beta^+_{-k\downarrow}\beta_{-k\downarrow})$$

The excitation spectrum: $E_k = \sqrt{\xi_k^2 + \Delta_k^2} = \sqrt{(\varepsilon_k - \mu)^2 + \Delta_k^2}$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}}\right), \quad v_k^2 = \frac{1}{2}\left[1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}}\right].$$

gap equation

$$\Delta_k = \frac{-1}{V} \sum_{k'} g(k,k') \langle a_{-k\downarrow} a_{k\uparrow} \rangle$$

$$a_{k\uparrow} = \cos\theta_k \alpha_{k\uparrow} - \sin\theta_k \beta^+_{-k\downarrow}, \quad a_{-k\downarrow} = \sin\theta_k \alpha^+_{k\uparrow} + \cos\theta_k \beta_{-k\downarrow}$$

$$\langle a_{-k\downarrow} a_{k\uparrow} \rangle = -\sin\theta_k \cos\theta_k \left[\langle \alpha^+_{k\uparrow}\alpha_{k\uparrow} \rangle - \langle \beta_{-k\downarrow}\beta^+_{-k\downarrow} \rangle \right]$$

$$= 2\sin\theta_k \cos 2\theta_k \left(1/2 - \langle \alpha^+_k \alpha_k \rangle + 1/2 - \langle \beta^+_k \beta^+_k \rangle \right\}/2$$

$$= \sin 2\theta_k \frac{1}{2} \tanh \frac{\beta E_k}{2} = \frac{\Delta_k}{2\sqrt{\xi_k^2 + \Delta_k^2}} \tanh \frac{\beta}{2}\sqrt{\xi_k^2 + \Delta_k^2}$$

$$\Rightarrow \boxed{\Delta_k = \int \frac{dk'}{(2\pi)^3} |g(k k')| \cdot \frac{\Delta_{k'}/2}{\sqrt{\xi_{k'}^2 + \Delta_{k'}^2}} \tanh \frac{\beta}{2} \sqrt{\xi_{k'}^2 + \Delta_{k'}^2}}$$ gap equation!

Let us consider the simplified version $|g(k,k')| = g$ $\Rightarrow$, Δ_k doesn't depend on k.

$$\Delta = gN(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon \quad \frac{\Delta}{2E_k}\tanh\frac{\beta}{2}E_k$$

i.e. $$\frac{1}{gN(0)} = \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon \quad \frac{1}{2E_k}\tanh\frac{\beta}{2}E_k$$

1° at $T = 0\,K$, $\Rightarrow$ $$\frac{1}{gN(0)} = \int_0^{\hbar\omega_D} d\epsilon_k \frac{1}{\sqrt{\epsilon_k^2+\Delta_k^2}} = \sinh^{-1}\frac{\hbar\omega_D}{\Delta}$$

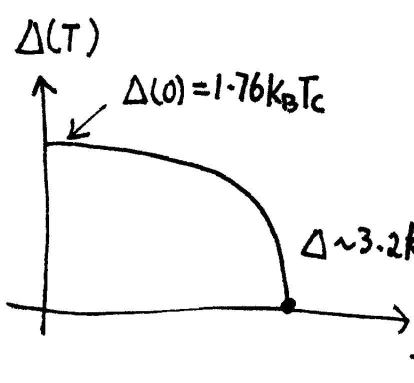
$$\Delta = \frac{\hbar\omega_D}{\sinh\frac{1}{N(0)g}} \sim 2\hbar\omega_D\, e^{-\frac{1}{N(0)g}} \quad \text{at } gN(0) \ll 1.$$

2° at T_c, where $\Delta \to 0$ $\Rightarrow$

$$\frac{1}{gN(0)} = \int_0^{\hbar\omega_D} d\epsilon \ \frac{1}{\epsilon}\tanh\frac{\beta_c\epsilon}{2} = \ln(1.14\,\beta_c\,\hbar\omega_D)$$

$$k_B T_c = 1.14\,\hbar\omega_D\, e^{-\frac{1}{N(0)g}}$$

$\Rightarrow$ $\Delta(T=0)/k_BT_c \approx 1.76$ for weak coupling BCS superconductor



≈ 2.3 Hg

2.15 Pb strong coupling superconductor

$3 \sim 12$ high T_c

§ Isotope effect.

The superconducting transition temperaturo $T_c = 1.14\ \hbar\omega_D\, e^{\frac{-1}{N(0)g}}$.

ω_D inversely dependens on $M_{ion}^{-1/2}$, which is a general result of the Newton equation $M\, d^2x/dt^2 = F$. $\Rightarrow T_c \propto M^{-1/2}$, which has been seen in Hg, Pb, Mg, Sn, Tl etc.

§ inclusion of Coulomb interaction

The effective electron-electron interaction contains two parts: a replusive Coulomb (screened) term, and a phonon term (attractive at $\omega < \omega_D$, repulsive at $\omega > \omega_D$).

– we neglect the $-g(k k')$'s dependence on the angle of $\vec{k}\cdot\vec{k}'$, but keep its frequency dependence. $-g(k k') = V_c - V_{ph}(\omega)$, $\omega = \frac{\xi_{\ell} - \xi_k}{\hbar}$

The gap equation

$$\Delta(\xi) = -N(0)\int d\xi' \left[V_c - V_{ph}(\omega)\right] \Delta(\xi')\ \frac{\tanh\frac{\beta\xi'}{2}}{2\xi'} \quad (\text{at } T_c)$$

$$= A + N(0)\int d\xi'\, V_{ph}\left[\frac{\xi-\xi'}{\hbar}\right] \Delta(\xi')\ \frac{\tanh\frac{\beta}{2}\xi'}{2\xi'}$$

where $A = -N(0)\int d\xi'\, V_c\, \Delta(\xi')\ \frac{\tanh\frac{\beta}{2}\xi'}{2\xi'}$ which is independent of ξ'.

At $\xi' \geq \hbar\omega_D$, the contribution is mainly from V_c, because $V_{ph}(\omega)$ is suppressed, and we set $\Delta(\xi) = A$ at $\xi > \hbar\omega_D$. We define the average value of Δ in the region of $\hbar\omega_D > |\xi|$ is B. The we have

$$B = N(0)\,\overline{V}_{ph} \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi'\, B \cdot \frac{\tanh\frac{\beta_c\xi'}{2}}{2\xi'} + A \approx B\,N(0)\,\overline{V}_{ph} \ln\frac{\hbar\omega_D}{k_B T_c} + A$$

where $\overline{V_{ph}}$ is kind of average value of $V_{ph}(\omega)$ at low frequency region. On the other hand

$$A = -N(0)\int d\xi' \; V_c \, \Delta(\xi') \frac{\tanh\frac{\beta}{2}\xi'}{2\xi'} = -N(0)V_c \left[B\int_{-\hbar\omega_D}^{\hbar\omega_D} + A\int_{\hbar\omega_D}^{\hbar\omega_c} + A\int_{-\hbar\omega_c}^{-\hbar\omega_D} \right]$$

$$\cdot d\xi' \frac{\tanh\frac{\beta_c}{2}\xi'}{2\xi'} \simeq -N(0)V_c \left[B \ln\frac{\hbar\omega_D}{k_B T_c} + A \ln\frac{\omega_c}{\omega_D} \right] \qquad (*)$$

where ω_c is a high-energy cut off for Coulomb interaction.

$$\Rightarrow \quad B\left(1 - N(0)\overline{V}_{ph} \ln\frac{\hbar\omega_D}{k_B T_c}\right) = A$$

$$A\left(1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}\right) = -N(0)V_c \left[\ln\frac{\hbar\omega_D}{k_B T_c}\right] B$$

$$\Rightarrow B\left(1 - N(0)\overline{V}_{ph} \ln\frac{\hbar\omega_D}{k_B T_c}\right) = -N(0)V_c \cdot \ln\frac{\hbar\omega_D}{k_B T_c} \Big/ \left(1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}\right) B$$

$$1 = \ln\frac{\hbar\omega_D}{k_B T_c}\left[N(0)\overline{V}_{ph} - \frac{N(0)V_c}{1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}}\right]$$

$$\Rightarrow \boxed{T_c = \hbar\omega_D \cdot \exp\left[- \frac{1}{N(0)\overline{V}_{ph} - N(0)V_c \big/ \left(1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}\right)} \right]}$$

effectively the phonon-interaction strength is weakened to

$$\overline{V}_{ph} \rightarrow \overline{V}_{ph} - V_c \Big/ \left(1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}\right).$$

* The Coulomb interaction is not very sufficient to suppress superconductivity, because of the renormalization effect in the denominator.

* a suppression of isotope effect $\dfrac{\delta T_c}{T_c} = \dfrac{\delta\omega_D}{\omega_D}\left[1 - \dfrac{N(0)V_c}{1 + N(0)V_c \ln\frac{\omega_c}{\omega_D}}\right]$

Lecture 12 Thermodynamic quantities in the SC state

§1 Condensation energy

$$H_{MF} = \sum_k \xi_k a^\dagger_{k\sigma} a_{k\sigma} - \Delta \sum_k (a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} + a_{-k\downarrow} a_{k\uparrow}) + Vol.\cdot \frac{\Delta^2}{g}$$

$$= \sum_k (a^\dagger_{k\uparrow}\ a_{-k\downarrow}) \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k & -\xi_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^\dagger_{-k\downarrow} \end{pmatrix} + \sum_k \xi_k + Vol.\cdot \frac{\Delta^2}{g}$$

$$= \sum_k E_k (\alpha^\dagger_{k\uparrow}\alpha_{k\uparrow} + \beta^\dagger_{-k\downarrow}\beta_{-k\downarrow}) + \sum_k (\xi_k - E_k) + Vol.\cdot\frac{\Delta^2}{g}$$

at $T=0 \Rightarrow \frac{E}{V} = \frac{1}{V}\sum_k (\xi_k - E_k) + \frac{\Delta^2}{g}$, because $\Delta = \frac{g}{2}\sum_k \frac{\Delta}{E_k}$

$$= \frac{1}{V}\sum_k (\xi_k - E_k + \frac{1}{2}\frac{\Delta^2}{E_k}) = \frac{1}{V}\sum_k [\xi_k - \frac{2\xi_k^2 + \Delta^2}{2E_k}]$$

$$= N(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon \ [\epsilon - \frac{2\epsilon^2+\Delta^2}{\sqrt{\epsilon^2+\Delta^2}}] = N(0)\{(\hbar\omega_D)^2\}\{1 - \sqrt{1+(\frac{\Delta}{\hbar\omega_D})^2}\}$$

$$\approx -\frac{1}{2} N(0)\Delta^2 \qquad (\hbar\omega_D >> \Delta)$$

only electrons are within $\pm\Delta$ of Fermi surface are significantly paired, each pair has a condenstation energy $\propto \Delta$. Each electron lowers the energy $\frac{\Delta}{E_F}\cdot\Delta$, which is a tiny number.

§2: specific heat

The Free energy $F(T) = -k_B T \ln Z$

$$= -k_B T \sum_k 2\ln [e^{-\frac{1}{2}\beta E_k} + e^{\frac{1}{2}\beta E_k}] + Vol.\cdot\frac{\Delta^2}{g}$$

$$= -k_B T \sum_k 2\ln[2\cosh \frac{\beta}{2} E_k] + Vol.\cdot \frac{\Delta^2}{g}$$

$$S = -\frac{\partial F}{\partial T} = k_B \sum_k 2\ln\left[2\cosh\frac{\beta}{2}E_k\right] + k_B T \sum_k 2\cdot\tanh\frac{\beta}{2}E_K \cdot \frac{\partial}{\partial T}\left[\frac{\beta E_K}{2}\right]$$

$$- \mathrm{Vol.}\ \frac{2\Delta}{g}\frac{\partial \Delta}{\partial T}$$

according to gap equation $\Delta = g\sum_K \tanh\frac{\beta}{2}E_K \frac{\Delta}{2E_K}$

$$S = k_B \sum_k 2\ln\left[2\cosh\frac{\beta}{2}E_k\right] - \sum_k 2\cdot\tanh\frac{\beta E_K}{2}\frac{E_K}{T} + \sum_K \tanh\frac{\beta}{2}E_K \frac{\Delta}{E_K}\frac{\partial\Delta}{\partial T}$$

$$- \sum_K \tanh\frac{\beta}{2}E_K \cdot \frac{\Delta}{E_K}\cdot\frac{\partial\Delta}{\partial T}$$

$$= 2k_B\left\{\sum_k \ln\left[2\cosh\frac{\beta}{2}E_k\right] - \sum_k \tanh\frac{\beta E_K}{2}\frac{\beta E_K}{2}\right\}$$

which can be represented as $S = -2k_B \sum_k (1-f_k)\ln(1-f_k) + f_k \ln f_k$

where $f_K = \frac{1}{e^{\beta E_K}+1}$

$$\Rightarrow C = T\frac{\partial S}{\partial T} = -\beta\frac{\partial S}{\partial\beta} = 2\beta k_B \sum_K \frac{\partial f_K}{\partial\beta}\ln\frac{f_K}{1-f_K} = -2\beta^2 k_B \sum_K E_k \frac{\partial f_K}{\partial\beta}$$

$$= -2\beta^2 k_B \sum_K E_K \cdot \frac{df_K}{d(\beta E_K)}\left[E_K + \beta\frac{dE_K}{d\beta}\right] \leftarrow \text{chain rule}$$

$$= 2\beta k_B \sum_K \left(-\frac{\partial f_K}{\partial E_K}\right)\Big(\underset{\substack{\uparrow\\ \text{Smooth}}}{E_K^2} + \frac{1}{2}\beta\frac{d\Delta^2(T)}{d\beta}\Big) \leftarrow \text{specific heat jump}$$

the first term at $T \to T_c$, where $\Delta \to 0$

$$2\beta k_B \sum_k \left(-\frac{\partial f}{\partial E_K}\right)E_K^2 = 2\beta k_B \sum_k \left(-\frac{\partial f}{\partial \xi_K}\right)\xi_k^2 = C_{\text{normal electron}}$$

$$= \frac{2\pi^2}{3}N_0 k_B^2 T \quad \text{which is a smooth term}$$

The jump term originates from the second term

$$\Delta C = (C_s - C_n)\Big|_{T=T_c} = N_0 \beta^2 k_B \frac{d\Delta^2}{d\beta} \int_{-\infty}^{+\infty} \left(-\frac{\partial f}{\partial \epsilon}\right) d\epsilon$$

$$= N(0) \left(-\frac{d\Delta^2}{dT}\right)\Big|_{T=T_c}$$

$$\left.\begin{array}{l} \\ \Delta(T) \approx 3.06\, k_B T_c \left(1 - \frac{T}{T_c}\right)^{1/2} \text{ around } T_c \end{array}\right\} \Rightarrow \frac{\Delta C}{C_N} \approx 1.43.$$

At low temperature, the temperature dependance of $\Delta(T)$ is negligible

$$C = 2\beta k_B \sum_k \left(-\frac{\partial f_k}{\partial E_k}\right) E_k^2 = 2\beta k_B \sum_k \frac{\beta e^{\beta E}}{(e^{\beta E}+1)^2} E_k^2$$

$$\approx 2\beta^2 k_B N(0) \int_{-\infty}^{+\infty} d\epsilon\; e^{-\beta E} \cdot E^2 \qquad e^{-\beta E} = e^{-\beta\sqrt{\Delta^2+\epsilon^2}} = e^{-\beta\Delta}\, e^{-\frac{\beta \epsilon^2}{2\Delta}}$$

$$\approx 2\beta^2 k_B N(0)\, e^{-\frac{\Delta(0)}{k_B T}}\, \Delta(0) \int_{-\infty}^{+\infty} d\epsilon\; e^{-\frac{\beta}{2}\frac{\epsilon^2}{\Delta}}$$

$$\approx 2N(0) \frac{\Delta^2(0)}{T} \left(\frac{2\pi\Delta(0)}{k_B T}\right)^{1/2} e^{-\frac{\Delta(0)}{k_B T}}$$

§ Spin susceptibility:

For the pair of $(k\uparrow, -k\downarrow)$, we have a 4D Hilbert space, with ground state $\Phi_k^{GP} = u_k |0_\uparrow 0_\downarrow\rangle - v_k |1_\uparrow 1_\downarrow\rangle$, and the excited pair state

$$\Phi_k^{ep} = v_k |0_\uparrow 0_\downarrow\rangle + u_k |1_\uparrow 1_\downarrow\rangle.$$

Breaking pair state $\Phi_k^{BP} = |0_\uparrow 1_\downarrow\rangle,\ |1_\uparrow 0_\downarrow\rangle$

It's clear to check that

$$\alpha_{k\uparrow}^{+} \Phi_k^{GP} = (\cos\theta_k a_{k\uparrow}^{+} + \sin\theta_k a_{-k\downarrow})(\cos\theta_k - \sin\theta_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle = |1_\uparrow 0_\downarrow\rangle$$

$$\beta_{-k\downarrow}^{+} \Phi_k^{GP} = (-\sin\theta_k a_{k\uparrow} + \cos\theta_k a_{-k\downarrow}^{+})(\cos\theta_k - \sin\theta_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle = |0_\uparrow 1_\downarrow\rangle$$

$\alpha^{+}_{k\uparrow} \beta^{+}_{-k\downarrow} \Phi^{GP}_{k} = v_k |0_\uparrow 0_\downarrow\rangle + u_k |1_\uparrow 1_\downarrow\rangle$, thus Φ^{BP}'s have the energy of E_k. and Φ^{EP} has the energy of $2E_k$. Now add an external magnetic field H, the BP states of $(0_\uparrow 1_\downarrow)$ and $(1_\uparrow 0_\downarrow)$ are shifted by $\mp \mu_B H$.

$$\Rightarrow M = \mu_B \sum_k \frac{e^{-\beta(E_k - \mu_B H)}}{(1+e^{-\beta E_k})^2} - \frac{e^{-\beta(E_k + \mu_B H)}}{(1+e^{-\beta E_k})^2}$$

(the numeritor is already at the linear order of H, thus we neglect the dependence on H in the denominator).

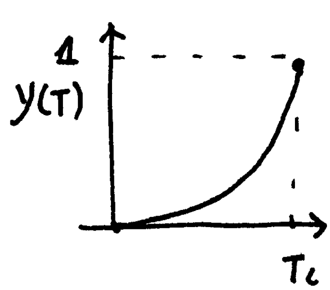
$$\chi = \frac{\partial M}{\partial H} = \mu_B^2 \sum_k \frac{e^{-\beta E_k}}{(1+e^{-\beta E_k})^2} \cdot (2\beta) = 2\beta \mu_B^2 \frac{N(0)}{2} \int d\epsilon \frac{1}{(e^{-\frac{\beta}{2}E} + e^{\frac{\beta}{2}E})^2}$$

$$= \mu_B^2 \frac{N(0)}{2} \int_{-\infty}^{\infty} \beta/2 \, \mathrm{sech}^2 \frac{\beta E}{2} \, d\epsilon \, , \quad \text{where } E = \sqrt{\epsilon^2 + \Delta^2}$$

$$\frac{\chi}{\chi_n} = \int_0^{+\infty} \frac{\beta}{2} \mathrm{sech}^2 \frac{\beta E}{2} \, d\epsilon = y\left(\frac{T}{T_c}\right) \leftarrow \text{Yoshida function}$$

at $T = T_c$, $\quad y(1) = \int_0^{+\infty} \mathrm{sech}^2 x \, dx = 1$

at low T, it's suppressed exponetially by $e^{-\beta\Delta}$ factor



normal density / superfluid density

Suppose the walls of the superconductor is moving with a small velocity v, and let the system at thermal equalibrium. The normal state density is defined as $\vec{P} = m\rho_n \vec{v}$. In other word, normal density is a susceptibility corresponding to a moving field $\vec{v}$, which result in $E_p \to E_p - \vec{v}\cdot\vec{p}$. How is it related to superfluid density?

According to London equation $J(r) = -\frac{\rho_s e^2}{mc} A(r)$, and

$$J(r) = \sum j_i = \sum_i \frac{e}{m}\left(P_i - \frac{eA(r_i)}{c}\right) = \frac{e}{m}\left(P - \frac{neA}{c}\right)$$ n is the total density

$$\Rightarrow \frac{e}{m} P = \frac{e}{m}\underbrace{\frac{eA}{c}}_{\to m\vec{v}}\rho_n \Rightarrow J(r) = \frac{e}{m}(n-\rho_n)\frac{eA}{c} \Rightarrow \rho_s + \rho_n = n.$$

The calculation of ρ_n: for the 4-states, $\phi_K^{GP} = u_K|0_\uparrow 0_\downarrow\rangle - v_K|1_\uparrow 1_\downarrow\rangle$, two of $\phi_K^{EP} = v_K|0_\uparrow 0_\downarrow\rangle + u_K|1_\uparrow 1_\downarrow\rangle$.

do not have energy shift. The two broken pair ~~state~~ has the energy shift $\pm \hbar\vec{k}\cdot\vec{v}$

$$\Rightarrow \vec{P} = \sum_k \hbar\vec{k}\left\{\frac{e^{-\beta(E_K - \hbar\vec{k}\cdot\vec{v})}}{(1+e^{-\beta E_K})^2} - \frac{e^{-\beta(E_K + \hbar\vec{k}\cdot\vec{v})}}{(1+e^{-\beta E_K})^2}\right\}$$

$$= \sum_k \hbar\vec{k}\,\frac{e^{-\beta E_K}}{(1+e^{-\beta E_K})^2}\,2\beta(\hbar\vec{k}\cdot\vec{v}) = \hbar^2\sum_k \vec{k}(\vec{k}\cdot\vec{v})\,\frac{\beta}{2}\,\mathrm{sech}^2\frac{\beta E_K}{2}$$

set $\vec{v}$ along the $\hat{z}$-axis, then $\vec{P}$ is also along $\hat{z}$-axis $\Rightarrow$

$$\vec{P} = \hbar^2 \sum_{k} k^2 v \cos^2\theta \, \frac{\beta}{2} \operatorname{sech}^2 \frac{\beta E_k}{2} = p_F^2 N(0) \int \frac{d\Omega}{4\pi} \cos^2\Theta \int_0^\infty d\epsilon \, \frac{\beta}{2} \operatorname{sech}^2 \frac{\beta E}{2}$$

$$= \frac{1}{3} p_F^2 [\, N(0)] \, Y(T) \Rightarrow P_n(T)/n = Y(T).$$

§ Compressbility and longitudinal field.

In the above two examples, the Cooper pair doesn't response, so that we have Yoshida type response. But in several cases, the Cooper pairs do response. For example, although the system has excitation gap, but the compressibility doesn't change much, because Cooper pair can response. Another example, where is the longitudinal $\vec{A}$ field, which can cause the condensate to flow, the Cooper pairs are made by electrons of $k + Q/2 \uparrow$ and $-k + Q/2 \downarrow$.

§ Coherent factors

Let us consider a general perturbation with the form of

$\Delta H = \sum_{k\sigma, k'\sigma'} B(k\sigma | k'\sigma') \, a^\dagger_{k\sigma} a_{k'\sigma'}$, we need to derive a general form for the transition rate.

$$a^\dagger_{k\sigma} = u_k^* \gamma^\dagger_{k\sigma} + \bar{\sigma} v_k^* \gamma_{-k\bar{\sigma}}$$

$$a_{k\sigma} = u_k \gamma_{k\sigma} + \bar{\sigma} v_k \gamma^\dagger_{-k\bar{\sigma}}$$

$$\Rightarrow \Delta H = \sum_{k\sigma, k'\sigma'} B(k\sigma | k'\sigma') \, [u_k^* \gamma^\dagger_{k\sigma} + \bar{\sigma} v_k^* \gamma_{-k\bar{\sigma}}][u_{k'} \gamma_{k'\sigma'} + \bar{\sigma}' v_{k'} \gamma^\dagger_{-k'\bar{\sigma}'}]$$

$$= \sum_{k\sigma, k'\sigma'} B(k\sigma|k'\sigma') \left[u_k^* u_{k'} \gamma_{k\sigma}^\dagger \gamma_{k'\sigma'} + \bar{\sigma}\bar{\sigma}' v_k^* v_{k'} \gamma_{-k\bar{\sigma}} \gamma_{-k'\bar{\sigma}'}^\dagger \right.$$

$$\left. + u_k^* v_{k'} \bar{\sigma}' \gamma_{k\sigma}^\dagger \gamma_{-k'\sigma'}^\dagger + \bar{\sigma} v_k^* u_{k'} \gamma_{-k\bar{\sigma}} \gamma_{k'\sigma} \right]$$

Let us consider the transition of scattering a quasiparticle from $k\sigma \to k'\sigma'$

$$\Rightarrow M(k\sigma|k'\sigma') = B(k\sigma|k'\sigma') \left[u_k^* u_{k'} \right] - \sigma\sigma' B(-k'\bar{\sigma}'|-k\bar{\sigma}) v_{-k'}^* v_{-k}$$

Let us assume $B(k\sigma|k'\sigma') = \eta\, \sigma\sigma' B(-k'\bar{\sigma}'|-k\bar{\sigma})$, where $\eta = \begin{cases} +1 & \text{case I} \\ -1 & \text{case II} \end{cases}$

$$\Rightarrow M(k\sigma|k'\sigma') = B(k\sigma|k'\sigma') \left[u_k^* u_{k'} - \eta v_{-k'}^* v_{-k} \right]$$

$\Rightarrow$ The transition rate $\dot{\nu} = \frac{2\pi}{\hbar} |M(k\sigma|k'\sigma')|^2 \left[f(E_{k'})(1-f(E_k)) \right.$

$$\left. - f(E_k)(1-f(E_{k'})) \right] \times \delta(\epsilon_k - \epsilon_{k'} - \hbar\omega)$$

$\Rightarrow$ power absorption

$$W = \sum_{kk'} \dot{\nu}\hbar\omega = 2\pi\omega B^2 \int_\Delta^\infty N_s(\epsilon) N_s(\epsilon') \, dE\, dE' \, (uu' - \eta vv')^2$$

$$\cdot \left[f(E') - f(E) \right] \delta(E - E' - \hbar\omega)$$

$\epsilon = \sqrt{E^2 - \Delta^2}$.

where $B^2 = \overline{|B(k\sigma|k'\sigma')|^2}$ is kind of an average.

$$N_s(\epsilon) = N(0) \frac{E}{\sqrt{E^2 - \Delta^2}} \quad (E > \Delta), \quad \left[N_s(E) = N(\epsilon) \cdot \frac{d\epsilon}{dE} = N(0) \frac{E}{\sqrt{E^2 - \Delta^2}} \right]$$

Jocobian

$$(uu' - \eta vv')^2 = \frac{1}{2} \left[1 + \frac{\xi\xi'}{EE'} - \eta \frac{\Delta^2}{EE'} \right]$$

$$\Rightarrow W = 4\pi\omega B^2 N(0) \int_\Delta^\infty dE \int_\Delta^\infty dE' \left[\frac{E}{\sqrt{E^2-\Delta^2}} \cdot \frac{E' - \eta\Delta^2}{\sqrt{E'^2-\Delta^2}} \right]$$

$$[f(\epsilon') - f(\epsilon)]\, \delta(\epsilon - \epsilon' - \hbar\omega).$$

The normalstate result can be obtained by setting $\Delta \to 0$.

$$\frac{W}{W_N} = \left[\int_\Delta^\infty dE \int_\Delta^\infty dE' \frac{E \cdot E' - \eta\Delta^2}{\sqrt{E^2-\Delta^2} \cdot \sqrt{E'^2-\Delta^2}} [f(E') - f(E)]\, \delta(E - E' - \hbar\omega) \right]$$

$$\cdot \left[\int_\Delta^\infty dE \int_\Delta^\infty dE' [f(E') - f(E)]\, \delta(E - E' - \hbar\omega) \right]^{-1}$$

The denominator $= \int_\Delta^\infty dE\, [f(E-\hbar\omega) - f(E)] = -\hbar\omega \int_\Delta^\infty dE \frac{\partial f}{\partial E} = \hbar\omega \frac{1}{1+e^{\beta\Delta}}$

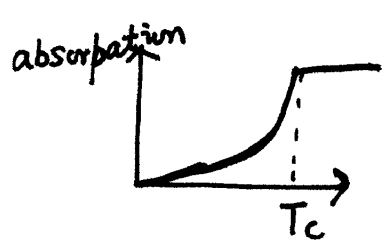
$\to \hbar\omega$ at $\Delta = 0$

$$\Rightarrow \boxed{\frac{W}{W_N} = \frac{1}{\hbar\omega} \int_\Delta^\infty dE \int_\Delta^\infty dE' \frac{EE' - \eta\Delta^2}{\sqrt{E^2-\Delta^2} \cdot \sqrt{E'^2-\Delta^2}} [f(E') - f(E)]\, \delta(E - E' - \hbar\omega)}$$

Example: ultra sonics

$$\Delta H_0 = \sum_{kk'\sigma} u\, V(k-k')\, a_{k\sigma}^{+} a_{k'\sigma'} \Rightarrow \text{type I}, \Rightarrow \eta = 1$$

$$\frac{W}{W_N} = \frac{1}{\hbar\omega} \int_\Delta^\infty dE \frac{E^2-\Delta^2}{E^2-\Delta^2} \frac{\partial f}{\partial E}(-\hbar\omega) = \frac{2}{1+e^{\beta\Delta}}$$



2) Nuclear relaxation: $\omega \to 0$

$$\Delta H = \sum J_{kk'} \, a^{+}_{k\uparrow} a_{k'\downarrow} + h.c. \qquad J_{kk'} = I_{k'-k}$$

because the flipping of spin, it belongs to type II, $\Rightarrow$

$$\frac{\omega}{\omega_N} = \int_{\Delta}^{\infty} dE \; \frac{E^2+\Delta^2}{E^2-\Delta^2} \left(-\frac{\partial f}{\partial E}\right)$$

$$= \frac{1/T_1}{1/T_{1n}}$$