Lect 5 — Spherical harmonics, CG coefficents

Consider a scaler wavefunction to represent the SO(3) group,

$$P_R \psi^{l,r}_m(x) = \psi^{l,r}_m(R^{-1}x) = \sum_{m'} \psi^{l,r}_{m'}(x)\, D^l_{m'm}(R)$$

where R is an SO(3) rotation, $m\ m'$ are labels of the $m(m')$-th basis function. $D^l_{m'm}(R)$ takes the form we derived before. r-represe[nts] another degree of freedom.

> HW: By using the Schwinger boson Rep, show that
>
> $$J_z|jm\rangle = m|jm\rangle, \quad J_\pm|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j\,m\pm1\rangle$$
>
> For SO(3), j and m take integer values. Then the matrix Rep
>
> $$(J_z)_{m'm} = m\delta_{m'm}, \quad (J_x)_{m'm} = \frac{1}{2}[\delta_{m'+1,m} f(m) + \delta_{m'-1,m}\, g(m)]$$
>
> $$(J_y)_{m'm} = \frac{1}{2i}[\delta_{m'+1,m} f(m) - \delta_{m'-1,m}\, g(m)]$$
>
> with $f(m) = \sqrt{(j-m)(j+m+1)}$, $g(m) = \sqrt{(j+m)(j-m+1)}$

P_R in the basis of $\psi^{l,r}_m(x)$, its matrix form is $D^l_{m'm}(R)$, and its generators I^l_a, which take the form of $J_{x,y,z}$ by setting $j=l$, and $-l \le m \le l$.

Now let's give the concrete expression of $\psi^l_m(x)$. Consider a single particle problem, and use spherical coordinate $x=(r,\theta,\varphi)$

Set a reference point $x_0 = (r, 0\ 0)$ along the z-axis. After $T = R(\varphi, \theta, \gamma)$, x_0 is rotated to $x = Tx_0$. Then

$$\psi^l_m(x) = \psi^l_m(Tx_0) = P_{T^{-1}}\psi^l_m(x_0) = \sum_{m'} \psi^l_{m'}(x_0) D^l_{m'm}(T^{-1})$$

$$= \sum_{m'} \psi^l_{m'}(x_0) \left[D^l_{mm'}(T)\right]^* = \sum_{m'} \psi^l_{m'}(x_0)\, e^{im\varphi} d^l_{mm'}(\theta)\, e^{im'\gamma}$$

Since the LHS is independent on γ, hence on RHS, $\psi^l_{m'}(x_0) = 0$ for all $m' \neq$

For $m' = 0$, we normalize by setting $\psi^l_m(x_0) = \delta_{m,0} \left(\frac{2l+1}{4\pi}\right)^{1/2} \phi_l(r)$,

Then $\psi^l_m(x) = \phi_l(r) \left(\frac{2l+1}{4\pi}\right)^{1/2} \underbrace{\left[D^l_{m,0}(\varphi, \theta, 0)\right]^*}$

The angular part is the spherical harmonics

$$Y_{lm}(\theta, \varphi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \left[D^l_{m,0}(\varphi, \theta, 0)\right]^*$$

$$= \left(\frac{2l+1}{4\pi}\right)^{1/2} e^{im\varphi}\, d^l_{m,0}(\theta)$$

Check normalization factor $\left(\frac{2l+1}{4\pi}\right)^{1/2}$:

$$\boxed{\delta_{ll'}\delta_{mm'} = \int_{-\pi}^{\pi} d\varphi \int_0^{\pi} d\theta\ \sin\theta\ Y^{l*}_m(\theta, \varphi)\, Y^{l'}_{m'}(\theta, \varphi).}$$

$$= \left(\frac{(2l+1)(2l'+1)}{4\pi}\right)^{1/2} \int_{-\pi}^{\pi} d\varphi \int_0^{\pi} d\theta\ \sin\theta\ D^l_{m0}(\varphi, \theta, 0)^* D^{l'}_{m'0}(\varphi, \theta) = \frac{\delta_{ll'}\delta_{mm'}}{2l+1} \frac{8\pi^2}{2\pi} \frac{2l+1}{4\pi}$$

$$= \delta_{ll'}\delta_{mm'}$$

$8\pi^2$ — SO(3)

2π for $\int_0^{2\pi} d\gamma$

According to $\boxed{d_{mm'}(\theta) = d_{m'm}(\theta)\,(-)^{m'-m}}$, we have

$$\left[Y_{lm}(\theta,\varphi)\right]^* = \left(\frac{2l+1}{4\pi}\right)^{1/2} e^{-im\varphi}\, d^l_{m0}(\theta,\varphi)$$

$$\Rightarrow \boxed{\left[Y_{lm}(\theta,\varphi)\right]^* = (-)^m\, Y_{l-m}(\theta,\varphi)}$$

We can further define Legendre function $P_l(\cos\theta) = d^l_{00}(\theta)$

Then $\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\gamma\; D^{l*}_{00}(\theta,\varphi,\gamma)\, D^{l'}_{00}(\theta\varphi,\gamma) = 4\pi^2 \int_0^{\pi} \sin\theta d\theta\; P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{\delta_{ll'}\, 8\pi^2}{2l+1}$

$$\Rightarrow \boxed{\int_0^{\pi} \sin\theta d\theta\; P_l(\cos\theta)\, P_{l'}(\cos\theta) = \frac{\delta_{ll'} \cdot 2}{2l+1}}$$

where $P_l(\cos\theta) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{l0}(\theta,$

* under inversion symmetry $\vec{x} \to -\vec{x}$: $\theta \to \pi-\theta$, $\varphi \to \pi+\varphi$

$$Y_{lm}(\pi-\theta, \pi+\varphi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} e^{-im(\pi+\varphi)}\, d^l_{m0}(\pi-\theta)$$

$\boxed{\text{Ex: prove } d^l_{mm'}(\pi-\theta) = (-)^{j+m}\, d^l_{m,-m'}(\theta) = (-)^{j-m'}\, d^l_{-m,m'}(\theta)}$

hence $d^l_{m0}(\pi-\theta) = (-)^{l+m}\, d^l_{m0}(\theta) \Rightarrow \boxed{Y_{lm}(\pi-\theta, \pi+\varphi) = (-)^l\, Y_{lm}(\theta,\varphi)}$

: Examples: $\sqrt{4\pi}\, Y_{00} = 1$, $\sqrt{\frac{4\pi}{3}}\, r\, Y^1_{\pm1} = \mp \frac{1}{\sqrt{2}}(x \pm iy)$, $\sqrt{\frac{4\pi}{3}}\, r\, Y^1_{10} = z$

$\sqrt{\frac{8\pi}{15}}\, r^2\, Y^2_{2\pm2} = \frac{1}{2}(x \pm iy)^2$, $\sqrt{\frac{8\pi}{15}}\, r^2\, Y^2_{2,\pm1} = \mp (x \pm iy) z$

$\sqrt{\frac{8\pi}{15}}\, r^2\, Y_{2,0} = \frac{1}{\sqrt{6}}(3z^2 - r^2)$

$\sqrt{\frac{8\pi}{35}}\, r^3\, Y_{3\pm3} = \mp \frac{1}{2\sqrt{2}}(x \pm iy)^3$, $\sqrt{\frac{8\pi}{35}}\, r^3\, Y_{3\pm2} = \frac{\sqrt{3}}{2}(x \pm iy)^2 z$, $\sqrt{\frac{8\pi}{35}}\, r^3\, Y_{3\pm1} = \mp \frac{\sqrt{30}}{20}(x \pm iy)(5z^2 - r^2)$

$\sqrt{\frac{8\pi}{35}}\, r^3\, Y_{30} = \frac{1}{\sqrt{10}}(5z^2 - 3r^2) z$

④

§ CG decomposition

Consider the product rep of two Reps $D^{j_1} \otimes D^{j_2} \simeq \oplus_j a_j D^j$. we can obtain a_j by using character function

$$\chi^{j_1}_{(\omega)} \otimes \chi^{j_2}_{(\omega)} = \sum a_j \chi^j(\omega).$$ Assume $j_1 \geq j_2$, then $\chi^j(\omega) = \dfrac{e^{i(j+1)\omega} - e^{-ij\omega}}{e^{i\omega}-1}$

Then $$\chi^{j_1}(\omega) \otimes \chi^{j_2}(\omega) = \sum_{\mu=-j_2}^{j_2} e^{i\mu\omega} \cdot \chi^{j_1}(\omega) = \sum_{\mu=-j_2}^{j_2} \frac{e^{i(j+\mu+1)\omega} - e^{-i(j+\mu)\omega}}{e^{i\omega}-1}$$

$$= \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega).$$ Hence the total $j = j_1+j_2, \cdots |j_1-j_2|$.

⊛ Define the CG coefficient $|j j_z\rangle = \sum_{m_1 m_2} |j_1 m_1\rangle |j_2 m_2\rangle \underline{\langle j_1 m_1 j_2 m_2 | j j_z\rangle}$

under the basis conventions

① $J_\pm |jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\, |j\, m\pm 1\rangle$ for both j_1 and j_2, and j

② $|j\, j_z = j\rangle = |j_1 j_1\rangle \otimes |j_2 j_2\rangle$, i.e. $\langle j_1 j_1 j_2 j_2 | j j\rangle = 1$ for $j = j_1+j_2$.
if $j = j_1+j_2$

③ when $j' = j \pm 1$, we impose $\langle j' M | J_{1z} | j M\rangle \geq 0$.

we can show that $\langle j_1 m_1 j_2 m_2 | j j_z\rangle$ is real. We have

$$\sum_{j j_z} \langle j_1 m_1 j_2 m_2 | j j_z\rangle \langle j_1 m_1' j_2 m_2' | j j_z\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | j j_z\rangle \langle j_1 m_1 j_2 m_2 | j j_z'\rangle = \delta_{j_z j_z'}$$

Some properties

HW: By using $J_\pm = J_{1\pm} + J_{2\pm}$ and $J_z = J_{1z} + J_{2z}$, prove that

① $\langle j_1 m_1 j_2 m_2 | j j_z \rangle$ is only nonzero, when $m_1 + m_2 = j_z$

② $$\sqrt{(j-j_z)(j+j_z+1)}\ \langle j_1 m_1 j_2 m_2 | j\, j_z+1 \rangle = \sqrt{(j_1+m_1)(j_1-m_1+1)}\ \langle j_1 m_1-1\, j_2 m_2 | j j_z \rangle + \sqrt{(j_2+m_2)(j_2 \mp m_2-1)}\ \langle j_1 m_1\, j_2 m_2-1 | j j_z \rangle$$

$$\sqrt{(j+j_z)(j-j_z+1)}\ \langle j_1 m_1 j_2 m_2 | j\, j_z-1 \rangle = \sqrt{(j_1-m_1)(j_1+m_1+1)}\ \langle j_1 m_1+1\, j_2 m_2 | j j_z \rangle + \sqrt{(j_2-m_2)(j_2+m_2+1)}\ \langle j_1 m_1\, j_2 m_2+1 | j j_z \rangle$$

The above recurisive formula only for C-G coefficient for the same $\vec{j}$. How to connect CG coefficients between different sector of j? We need to use the difference of angular momenta $\frac{1}{2}(j_{1z} - j_{2z}) = -\frac{1}{2}(j_{1z}+j_{2z}) + j_{1z}$

Since $j_{1z}+j_{2z}$ does not chang j, only j_{1z} contributes.

Actually $\left.\begin{matrix}\vec{j}_1-\vec{j}_2 \\ \vec{j}_1+\vec{j}_2\end{matrix}\right\}$ belong to SO(4) generators, ~~which~~ and $\vec{j}_1-\vec{j}_2$ can only change the total angular momenta $\vec{j}_1+\vec{j}_2$ by 1.

We have $\quad j_{1z} |j_1 m_1\rangle \otimes |j_2 m_2\rangle = m_1 |j_1 m_1\rangle |j_2 m_2\rangle$

$$\Rightarrow \hat{j}_{1z} \sum_{j j_z} |j j_z\rangle \langle j j_z | j_1 m_1 j_2 m_2 \rangle = m_1 \sum_{j j_z} |j j_z\rangle \langle j j_z | j_1 m_1 j_2 m_2 \rangle$$

$$\Rightarrow \sum_{j'} \langle j j_z | \hat{j}_{1,z} | j' j_z \rangle \langle j' j_z | j_1 m_1 j_2 m_2 \rangle = m_1 \langle j j_z | j_1 m_1 j_2 m_2 \rangle$$

only nonzero if $j' = j \pm 1$.

$$\Rightarrow \langle j j_z | \hat{j}_{1z} | j+1\, j_z \rangle \langle j+1\, j_z | j_1 m_1 j_2 m_2 \rangle + \langle j j_z | \hat{j}_{1z} | j-1, j_z \rangle \langle j-1\, j_z | j_1 m_1 j_2 m_2 \rangle = m_1 \langle j j_z | j_1 m_1 j_2 m_2 \rangle$$

By the above relation, and the recursive relations,

we can derive $\langle j_1 m_1 j_2 m_2 | j j_z \rangle$ by iterations.

§ Symmetry properties of CG coefficients

① If we switch the sequence of $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$, we will have

$$|j\, j_z^{(21)}\rangle = \sum |j_2 m_2\rangle \otimes |j_1 m_1\rangle \langle j_2 m_2 j_1 m_1 | j j_z\rangle$$

Hence $\langle j_1 m_1 j_2 m_2 | j j_z\rangle$ and $\langle j_2 m_2 j_1 m_1 | j j_z\rangle$ can only differ by a constant. Except the convention of $\langle j' M^{(12)} | j_{1z} | j M^{(12)}\rangle \geq 0$ for $\langle j_1 m_1 j_2 m_2 | j j_z\rangle$ the other two conditions are the same. We should replace it as

$$\langle j' M^{(21)} | j_{2z} | j m^{(21)}\rangle \geq 0, \quad \text{since} \langle j' M | j_{1z} + j_{2z} | j M\rangle = 0.$$

it means $\langle j' M^{(21)} | j_{1z} | j m^{(21)}\rangle \leq 0$, For example $\langle \bar{j}-1\, M^{(21)} | j_{1z} | \bar{j} M^{(21)}\rangle \leq$
with $\bar{J} = j_1 + j_2$

Since $|\bar{j}\bar{j}\rangle^{(12)} = |\bar{j}\bar{j}\rangle^{(21)}$, by appling j_-, we have

$|\bar{j} M\rangle^{(21)} = |\bar{j} M\rangle^{(12)}$. But the condition $\langle \bar{j}-1 M^{(21)} | j_{1z} | \bar{j} M^{(21)}\rangle \leq$

means that $\boxed{|\bar{j}-1, M^{(21)}\rangle = (-)\, |\bar{j}-1, M\rangle^{(12)}}$. Similarly, we

have $|\bar{j}-k, M^{(21)}\rangle = (-)^k\, |\bar{j}-1, M\rangle^{(12)}$, hence

$$\boxed{\langle j_1 m_1 j_2 m_2 | j m\rangle = (-)^{j_1+j_2-j} \langle j_2 m_2 j_1 m_1 | j m\rangle}$$

The CG coefficient has more symmetries, since j_1, j_2 and j for a triangle, we can also use j_2 and j to form j_1, and $j\,j_1 \to j_2$.

Wigner introduced 3j symbol

$$\boxed{\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \frac{(-)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 | j_3 (-m_3)\rangle \\ \text{and } \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle &= (-)^{-j_1+j_2-m_3} \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \end{aligned}}$$

Then
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

$$= (-)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

$$= (-)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

see https://en.wikipedia.org/wiki/3-j_symbol

$$\sum_{j_3 m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{j_3 j_3'} \delta_{m_3 m_3'}}{2j_3+1}$$

when $j_1\, j_2\, j_3$ satisfy $\triangle$ relation

⊛ Analytic calculation of CG coefficients

HW: Start with $|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{jm} |jm\rangle\langle jm|j_1 m_1 j_2 m_2\rangle$,

Apply rotation $D(R)$ to both side, prove that

$$D^{j_1}_{m_1 m_1'}(R)\, D^{j_2}_{m_2 m_2'}(R) = \sum_j \langle j_1 m_1 j_2 m_2|jm\rangle\langle j_1 m_1' j_2 m_2'|jm'\rangle\, D^j_{mm'}(R).$$

set $\alpha = \gamma = 0$, we have

$$d^{j_1}_{m_1' m_1}(\beta)\, d^{j_2}_{m_2' m_2}(\beta) = \sum_j \langle j_1 m_1 j_2 m_2|jm\rangle\langle j_1 m_1' j_2 m_2'|jm'\rangle\, d^j_{m'm}(\beta)$$

use the relation $\int_0^\pi d\beta \sin\beta\, d^{j_1}_{m_1' m_1}(\beta)\, d^{j_2}_{m_2' m_2}(\beta) = \frac{2}{2j_1+1}\delta_{j_1 j_2}$

Hence

$$\langle j_1 m_1 j_2 m_2|jm\rangle\langle j_1 m_1' j_2 m_2'|jm'\rangle = \frac{2j+1}{2}\int_0^\pi d\beta \sin\beta\, d^j_{m'm}(\beta)\, d^{j_1}_{m_1' m_1}(\beta)\, d^{j_2}_{m_2' m_2}(\beta)$$

with $m = m_1 + m_2$, $m' = m_1' + m_2'$.

$$d^j_{m'm}(\beta) = \sum_n (-)^n \frac{\{(j+m)!(j-m)!\,(j+m')!\,(j-m')!\}^{1/2}}{(j+m'-n)!\,(j-m-n)!\,n!\,(n-m'+m)!}\left(\cos\frac{\beta}{2}\right)^{2j+m'-m-2n}\left(\sin\frac{\beta}{2}\right)^{2n-m'+m}$$

← the same expression as in previous note
Change of dummy variable.

Set $m_1 = j$ and $m_2 = -k$

$$d^{j_1}_{m_1' j_1} = \sum_n (-)^n \frac{[(j_1+m_1')!\,(j_1-m_1')!\;2j!]^{1/2}}{[j_1+m_1'-n]!\,(-n)!\,n!\,(n-m_1'+j_1)!}\left(\cos\frac{\beta}{2}\right)^{j+m'-2n}\left(\sin\frac{\beta}{2}\right)^{2n-m'+\dots}$$

$\boxed{n=0}$ →

$$= \frac{(2j_1!)^{1/2}}{[(j_1+m_1')!\,(j_1-m_1')!]^{1/2}}\left(\cos\frac{\beta}{2}\right)^{j_1+m_1'}\left(\sin\frac{\beta}{2}\right)^{j_1-m_1'}$$

$$d^{j_2}_{m_2',-j_2} = \sum_n (-)^n \frac{\left[(j_2+m_2')!\,(j_2-m_2')!\,(2j_2)!\right]^{1/2}}{\underbrace{(j_2+m_2'-n)!}\,(j_2-(-j_2)-n)!\,n!\,\underbrace{(n-j_2-m_2')!}}$$

$$\left(\cos\frac{\beta}{2}\right)^{3j_2+m_2'-2n}\left(\sin\frac{\beta}{2}\right)^{2n-m_2'-j_2} \qquad \Rightarrow n = j_2+m_2'$$

$$= (-)^{j_2+m_2'}\frac{\left[(j_2+m_2')!\,(j_2-m_2')!\,(2j_2)!\right]^{1/2}}{(j_2-m_2')!\,(j_2+m_2')!}\left(\cos\frac{\beta}{2}\right)^{j_2-m_2'}\left(\sin\frac{\beta}{2}\right)^{j_2+m_2'}$$

$$= (-)^{j_2+m_2'}\left[\frac{(2j_2)!}{(j_2-m_2')!\,(j_2+m_2')!}\right]^{1/2}\left(\cos\frac{\beta}{2}\right)^{j_2-m_2'}\left(\sin\frac{\beta}{2}\right)^{j_2+m_2'}$$

we have integral $\frac{1}{2}\int_0^\pi d\beta\,\left(\cos\frac{\beta}{2}\right)^{2a}\left(\sin\frac{\beta}{2}\right)^{2b}\sin\beta = \frac{a!\,b!}{(a+b+1)!}$

$$\int_0^\pi \underbrace{d\beta}_{\sin\beta}\left(\cos\frac{\beta}{2}\right)^{j_1+j_2+m_1'-m_2'+2j+\underbrace{m'-m}_{(m_1'+m_2'-j_1+j_2)}-2n}\left(\sin\frac{\beta}{2}\right)^{j_1+j_2-m_1'+m_2'+2n-(m_1'+m_2')+(j_1-j_2)}$$

$$= \int_0^\pi d\beta\,\sin\beta\left(\cos\frac{\beta}{2}\right)^{2j+2j_2+2m_1'-2n}\left(\sin\frac{\beta}{2}\right)^{2j_1-2m_1'+2n} \qquad \begin{aligned} m' &= m_1'+m_2' \\ m &= j_1-j_2 \end{aligned}$$

$$= \frac{2\cdot(j+j_2+m_1'-n)!\,(j_1-m_1'+n)!}{(j_1+j_2+j+1)!}$$

Hence $\langle j_1m_1'j_2m_2'|jm\rangle\langle j_1j_1\,j_2{-j_2}|j\,j_1{-j_2}\rangle = \frac{2j+1}{2}\int_0^\pi \sin\beta\,d\beta\; d^{j_1}_{m_1'j_1}\,d^{j_2}_{m_2'-j_2}\,d^{j}_{m'm}$

$$= (2j+1)(-)^{j_2+m_2'}\frac{\left[(2j_1)!\,(2j_2)!\right]^{1/2}\left[(j+j_1-j_2)!\,(j-j_1+j_2)!\right]^{1/2}}{\left[(j_1+m_1')!\,(j_1-m_1')!\,(j_2+m_2')!\,(j_2-m_2')!\right]^{1/2}(j_1+j_2+j+1)!}$$

$$\cdot\sum_n (-)^n\frac{\left[(j+m_1'+m_2')!\,(j-m_1'-m_2')!\right]^{1/2}(j+j_2+m_1'-n)!\,(j_1-m_1'+n)!}{(j+m_1'+m_2'-n)!\,(j-j_1+j_2-n)!\,n!\,(n-m_1'-m_2'+j_1-j_2)!}$$

$$= \frac{(2j+1)\{(2j_1)!(2j_2)!\}^{1/2}}{(j_1+j_2+j+1)!} \left[\frac{(j+j_1-j_2)!(j-j_1+j_2)!\,(j+m_1'+m_2')(j-m_1'-m_2')}{(j_1+m_1')!\,(j_1-m_1')!\,(j_2+m_2')!\,(j_2-m_2')!}\right]^{1/2}.$$

$$\cdot \sum_n (-)^{n+j_2+m_2'} \frac{(j+j_2+m_1'-n)!\,(j_1-m_1'+n)!}{(j+m_1'+m_2'-n)!\,(j-j_1+j_2-n)!\,n!\,(n+j_1-j_2-m_1'-m_2')!}.$$

Set $m_1'=j_1,\ m_2'=-j_2 \Rightarrow (\langle j_1 j_1 j_2 -j_2 | j, j_1-j_2\rangle)^2$

$$= \frac{(2j+1)[(2j_1)!(2j_2)!]^{1/2}}{(j+j_1+j_2+1)!} \frac{[(j+j_1-j_2)!(j-j_1+j_2)!]}{[(2j_1)!(2j_2)!]^{1/2}} \cdot \sum_n (-)^n \frac{(j+j_1+j_2-n)!}{n!\,(j+j_1-j_2-n)!\,(j-j_1+j_2-n)!}$$

$$= \frac{(2j+1)(j+j_1-j_2)!\,(2j_2)!}{(j+j_1+j_2+1)!} \sum_n (-)^n \frac{(j-j_1+j_2)!}{n!\,(j-j_1+j_2-n)!} \frac{(j+j_1+j_2-n)!}{(2j_2)!\,(j+j_1-j_2-n)!}$$

$$= \frac{(2j+1)(2j_2)!\,(j+j_1-j_2)!}{(j+j_1+j_2+1)!} \sum_n (-)^n \binom{j-j_1+j_2}{n}\binom{j+j_1+j_2-n}{j+j_1-j_2-n}$$

According to $\sum_p (-)^p \binom{u}{p}\binom{v-p}{r-p} = \binom{v-u}{r} \leftarrow$ $n=p$, $u=j-j_1+j_2$, $v=j+j_1+j_2$, $r=j+j_1-j_2$

$\Rightarrow |\langle j_1 j_1 j_2 -j_2 | j, j_1-j_2\rangle|^2$

$$= \frac{(2j+1)(2j_2)!\,(j+j_1-j_2)!}{(j+j_1+j_2+1)!} \binom{2j_1}{j+j_1-j_2} = \frac{(2j+1)(2j_1)!\,(2j_2)!}{(j+j_1+j_2+1)!\,(j_1+j_2-j)!}$$

It can be proved that $\langle j_1 j_1 j_2 -j_2 | j\ j_1-j_2\rangle > 0 \Rightarrow$

$$\langle j_1 j_1 j_2 -j_2 | j\ j_1-j_2\rangle = \left[\frac{(2j+1)(2j_1)!\,(2j_2)!}{(j+j_1+j_2+1)!\,(j_1+j_2-j)!}\right]^{1/2}$$

Then

$$\langle j_1 m_1' j_2 m_2' | jm\rangle = \frac{(2j+1)^{1/2}(j_1+j_2-j)^{1/2}}{[j+j_1+j_2+1]!^{1/2}} \left(\frac{(j+j_1-j_2)!\,(j-j_1+j_2)!\,(j+m_1'+m_2')(j-m_1'-m_2'}{(j_1+m_1')!(j_1-m_1')!(j_2+m_2')!(j_2-m_2')!} \right.$$

$$\cdot \sum_n (\cdots\cdots)$$

$$= \left\{ \frac{(j_1+j_2-j)!\,(j-j_1+j_2)!\,(j+j_2-j_1)!}{[j+j_1+j_2+1]!} \right\}^{1/2} \left\{ \frac{(2j+1)\,(j+m_1'+m_2')!(j-m_1'-m_2')!}{(j_1+m_1')!\,(j_1-m_1')!\,(j_2+m_2')!\,(j_2-m_2')!} \right\}^{1/2}$$

$$\sum_n (-)^{n+j_2+m_2'} \frac{(j+j_2+m_1'-n)!\ \ (j_1-m_1'+n)!}{(j+m_1'+m_2'-n)!\ (j-j_1+j_2-n)!\ n!\ (n+j_1-j_2-m_1'-m_2')!}$$

$(-)^{n+j_2+m_2'} = (-)^{n-j_2-m_2'}$ since j_2+m_2' is integer.

Simplify notation $m'_{1,2} \to m_{1,2}$, and $n-j_2-m_2 \to n$ or $n \to n+j_2+m$

Define $\Delta(j_1 j_2 j) = \left\{ \frac{(j_1+j_2-j)!\ (j+j_1-j_2)!\ (j+j_2-j_1)!}{(j+j_1+j_2+1)!} \right\}^{1/2}$

$$\boxed{\langle j_1 m_1 j_2 m_2 | jm\rangle = \Delta(j_1 j_2 j) \left[\frac{(2j+1)^{1/2}(j+m_1+m_2)!\,(j-m_1-m_2)!}{(j_1+m_1')!\,(j_1-m_1')!\,(j_2+m_2')!\,(j_2-m_2')!} \right]^{1/2} \cdot \sum_n (-)^n \frac{(j+m_1-m_2-n)!\ \ (j_1+j_2-m_1+m_2+n)!}{[\,j-j_1-m_2-n)!\,(j-j_2+m_1-n)!\ (n+j_2+m_2)!\ (n+j_1-m_1)!}}$$

CG coefficient

$$\max(-j_2-m_2,\ -j_1+m_1) \le n \le \min[j_1-j_1-m_2,\ j-j_2+m_1]$$