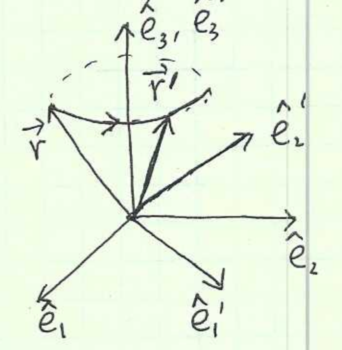
Chapter 1 3D rotation group: SO(3) v.s SU(2)

§: Consider a 3d space with the coordinate basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, and a vecter $\vec{r} = (\hat{e}_1, \hat{e}_2, \hat{e}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_i \hat{e}_i x_i$.



Consider a rotation that rotates

$$\vec{r} \to \vec{r}' = (\hat{e}_1, \hat{e}_2, \hat{e}_3)\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

where $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, where R is a 3×3 real matrix.

If we require that for any vector $\vec{r}$, after such a rotation

$$\vec{r}'\cdot\vec{r}' = \vec{r}\cdot\vec{r} \Rightarrow (x_1'\ x_2'\ x_3')\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = (x_1\, x_2\, x_3)\, R^T R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1\, x_2\, x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

EX: Prove that $\boxed{R^T R = I_{3\times 3}}$.

This defines the orthogonal matrix $O(3)$. Since $\det R^T = \det R$, we have $\det R = \pm 1$. $O(3)$ can be devided into two cosets; the part with $\det R = 1$, and that with $\det R = -1$. These two cosets are disconneted. The former part forms a group denoted as $SO(3)$ – specical orthogonal group, which can be continously conneted with the identity operation. – basically, it's formed by rotation. The other coset is represented as $\sigma \cdot SO(3)$, or $I \cdot SO(3)$, where σ is a reflection, I is spacial inversion.

$\boxed{O(3)/SO(3) = Z_2}$

$SO(3) \leftarrow \det = 1$

$\sigma \cdot SO(3) \leftarrow \det = -1$

Comments:

① Two viewpoints for rotations (coordinate transformations)

$$\vec{r}\,' = (\hat{e}_1\ \hat{e}_2\ \hat{e}_3)\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = (\hat{e}_1', \hat{e}_2', \hat{e}_3')\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $\boxed{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$, and $\boxed{(\hat{e}_1', \hat{e}_2', \hat{e}_3') = (\hat{e}_1\ \hat{e}_2\ \hat{e}_3)R}$

initiative: rotation of vector

passive: rotation of reference frame.

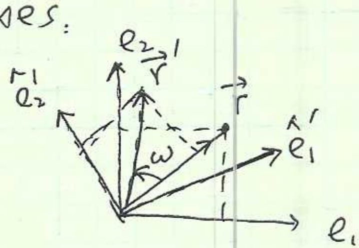
② reflection: with respect to xy-plane $\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$: $\det \sigma = -1$.

inversion I: $\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$: $\det I = -1$.

reflection and inversion are different up to a rotation.

§ Rotations around $\hat{e}_1, \hat{e}_2, \hat{e}_3$ axes.

$$R(\hat{e}_3, \omega) = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$= \exp\{-i\omega T_3\} \quad \text{where } T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex: Prove the above expression.

Hint: We only need to consider the 2×2 diagonal block, then $T_3 \sim \sigma_2$

$$\exp\{-i\omega\,\sigma_2\} = \cos\omega - i\sigma_2 \sin\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}.$$

Similarly, we have

$$R(\vec{e}_1, \omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix} = \exp[-i\omega T_1] \quad \text{with } T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$R(\vec{e}_2, \omega) = \begin{pmatrix} \cos\omega & & \sin\omega \\ & 1 & \\ -\sin\omega & & \cos\omega \end{pmatrix} = \exp[-i\omega T_2] \quad \text{with } T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

The SO(3) generators T_1, T_2, T_3 can be represented as

$$\boxed{(T_a)_{bc} = -i\,\epsilon_{abc}}$$

$\boxed{\text{EX:}}$ Check the commutaters $[T_a, T_b] = i\,\epsilon_{abc} T_c$, i.e.

T's are angular momentum operaters, Actually in the spin-1, representation.

How about rotation around an arbitary axes $\hat{n} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$.

Define a rotation $S(\varphi, \theta)$, which rotates $\hat{e}_3 \to \hat{n}$.

$$S(\varphi, \theta) = R(\hat{e}_3, \varphi) R(\hat{e}_2, \theta) = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\cos\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R[\hat{n}, \omega] = S(\varphi, \theta) R(\hat{e}_3, \omega) S^{-1}(\varphi, \theta) = S \exp[-i\omega T_3] S^{-1}$$

$$= \exp[-i\omega\, S T_3 S^{-1}]$$

HW: Prob 1: Prove that by using $S(\varphi, \theta)$ defined above

$$S T_3 S^{-1} = n_1 T_1 + n_2 T_2 + n_3 T_3 = \hat{n} \cdot \vec{T}$$

where $\vec{T} = \hat{e}_1 T_1 + \hat{e}_2 T_2 + \hat{e}_3 T_3$

Hence

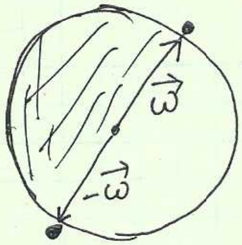
$$R[\hat{n}, \omega] = \exp[-i\omega\, \hat{n}\cdot\vec{T}\,] = \exp[-i\,\vec{\omega}\cdot\vec{T}\,]$$

where
$$\omega_1 = \omega n_1 = \omega \sin\theta\cos\varphi$$
$$\omega_2 = \omega n_2 = \omega \sin\theta \sin\varphi$$
$$\omega_3 = \omega n_3 = \omega\cos\theta \,.$$

Due to the relation $R[\hat{n}, \omega] = R[-\hat{n}, 2\pi - \omega]$,

The parameter $\vec{\omega}$, lives in the solid ball with radius π.

and $R(\hat{n}, \pi) = R(-\hat{n}, \pi)$. $\Rightarrow$ ~~the surface~~ diameter ends on the surface are identified.



$\leftarrow$ Solid ball for SO(3). $\quad 0 \le \omega \le \pi$

☆ all the rotations with the same rotation angle are in the same conjugacy class. its trace (character) is $1 + 2\cos\omega$.

$$\boxed{\begin{array}{l}\text{HW: Prob 2:} \\ \text{Write down the } 3\times 3 \text{ matrix for } R[\hat{n}, \omega],\ \text{with } \hat{n} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}.\end{array}}$$

§ Parameterization of 3D rotations

When we write $R[\hat{n}, \omega] = e^{-i\vec{\omega}\cdot\vec{T}}$, the rotation axis is along a general direction, hence is not convenient. It will be nice if any rotation, can be decomposed into a series of rotation around a set of fixed axes. Eulerian angles can do this. Since you have learned this in Classic mechanics and QM, I will not draw figures. Any rotation R can be decomposed into

$$\boxed{R(\alpha, \beta, \gamma) = R(\hat{z}, \alpha)\, R(\hat{y}, \beta)\, R(\hat{z}, \gamma)}$$

- Rotate around $\hat{z}$-axis at angle α
- Rotate around the new position of $\hat{y}$-axis at the angle of β $\hat{y}'$,
- Rotate around the current position of $\hat{z}$-axis, at the angle of γ.

HW. Problem 3.

Work out the 3x3 matrix by using the Eulerian angles α, β, γ

$$R(\alpha, \beta, \gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha, & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha, & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha, & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha, & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta, & \sin\gamma\sin\beta, & \cos\beta \end{bmatrix}$$

§ SU(2)

Consider a 2x2 traceless Hermitian matrix X, which can be expanded as

$$X = x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 = \vec{\sigma}\cdot\vec{r} = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix}$$

$$x_i = \tfrac{1}{2} \mathrm{tr}[X \sigma_i], \quad \det X = -(x_1^2 + x_2^2 + x_3^2).$$

Hence, X and $\vec{r}$ have a one-to-one correspondence.

$$\boxed{\text{EX: prove that } (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + \vec{\sigma}\cdot(\vec{a}\times\vec{b}).}$$

Consider a rotation $\vec{r} \to \vec{r}'$: $x'_a = R_{ab} x_b$, then

$$X' = x'_a \sigma_a = R_{ab} x_b \sigma_a = x_b \sigma'_b \quad \text{with } \sigma'_b = \sigma_a R_{ab}$$

We define an unitary transformation $U^\dagger = U^{-1}$.

$$\boxed{U \sigma_b U^{-1} = \sum_{a=1}^{3} \sigma_a R_{ab}}$$

This defines a mapping between U and R.

The mapping from $U \to R$ is <u>two to one</u>, since $\pm U$ lead to the same R.

Denote

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SU(2) \Rightarrow U^\dagger U = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & a^*c + b^*d \\ \text{c.c} & |c|^2+|d|^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} |a|^2+|b|^2 = |c|^2+|d|^2 = 1 \\ a^*c + b^*d = 0 \\ \det = ad - bc = 1 \end{cases} \Rightarrow |a-d^*|^2 + |b+c^*|^2 = \left[|a|^2+|b|^2+|c|^2+|d|^2\right] - \left[\det + \det^*\right] = 0$$

$$\Rightarrow U = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \quad \text{and} \quad \boxed{\begin{aligned} a &= h_0 - i h_3 \\ b &= h_2 - i h_1 \end{aligned} \quad \text{with } h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1}$$

This means that $\boxed{\text{the group space of } SU(2) \text{ is } S^3!}$

(*) Relation to spin-1/2 in B-field

starting with $\sigma_z \cdot \chi(\hat{z}) = +1\chi(\hat{z})$ where $\chi(\hat{z}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mathcal{U}$ defines a rotation operater in the spiner space

$\mathcal{U}[\sigma_z \mathcal{U}^{-1}]\left(\mathcal{U}\chi(\hat{z})\right) = \mathcal{U}\chi(\hat{z})$, then $\mathcal{U}\chi(\hat{z})$ is the eigenstate with eigenvalue 1 of $\mathcal{U}\sigma_z\mathcal{U}^{-1}$.

Using Eulerian angules

$$\mathcal{U} = e^{-i\frac{\sigma_z}{2}\alpha}\, e^{-i\frac{\sigma_y}{2}\beta}\, e^{-i\frac{\sigma_z}{2}\gamma} = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}$$

$$= \begin{bmatrix} e^{-i\frac{(\gamma+\alpha)}{2}}\cos\frac{\beta}{2}, & -e^{i\frac{\gamma-\alpha}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{-\gamma+\alpha}{2}}\sin\frac{\beta}{2}, & e^{i\frac{\gamma+\alpha}{2}}\cos\frac{\beta}{2} \end{bmatrix}$$

and $\chi(\hat{n}) = e^{-i\frac{\gamma+\alpha}{2}}\begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2}e^{i\alpha} \end{bmatrix}$, where $\hat{n} = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix}$

HW problem:

Prove that

$$u(\alpha\beta\gamma)\,\sigma^a\,u^{-1}(\alpha\beta\gamma) = R_{ab}\,\sigma^b$$

with

$$u(\alpha\beta\gamma) = \begin{bmatrix} e^{-i\frac{\gamma}{2}-i\frac{\alpha}{2}}\cos\frac{\beta}{2} & -e^{i\frac{\gamma}{2}-i\frac{\alpha}{2}}\sin\frac{\beta}{2} \\ e^{-i\frac{\gamma}{2}+i\frac{\alpha}{2}}\sin\frac{\beta}{2} & e^{i\frac{\gamma}{2}+i\frac{\alpha}{2}}\cos\frac{\beta}{2} \end{bmatrix}$$

and

$$R(\alpha\beta\gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha, & -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha, & \sin\beta\cos\alpha \\ \cos\gamma\cos\beta\sin + \sin\gamma\cos\alpha, & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha, & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta, & \sin\gamma\sin\beta, & \cos\beta \end{bmatrix}$$

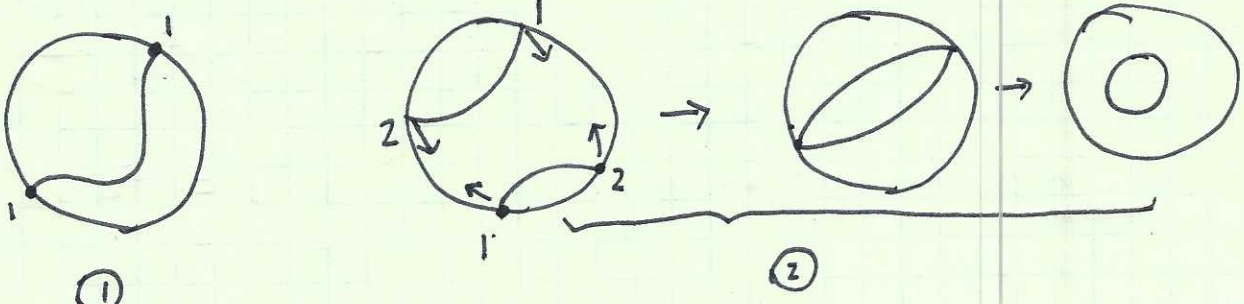
§ The global properties of SO(3) and SU(2)

We have mentined that the group space of SO(3) is a solid radius π.
ball with its surface identified as one point.
any two ends of a diameter on

The the closed loops in SO(3) can be classified into two classes.

① A loop passing the surface odd number of times

② . . . . even number of times



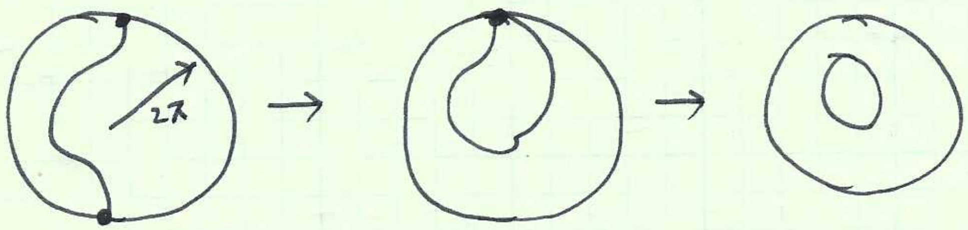
①   ②

Hence, SO(3) group space is doubly connected!

(or, more formally, the fundmental group is Z_2).

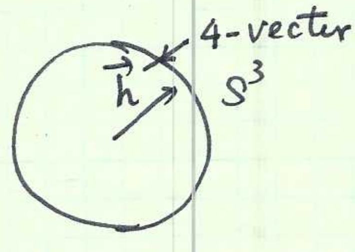
• For SU(2) $\quad u = e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\,\omega}$

Then it's group space is a ball with a radius of 2π, and all points on the surface are identical $\rightarrow -1$. Then its group space is singly connected. It's fundmental group is trivial.



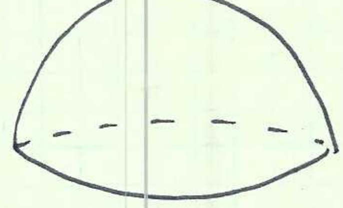
• This conclusion can also be drawn from another mapping

① SU(2) group space is S^3-sphere, certainly simple connected.



② due to $-h_1, -h_2, -h_3, -h_4$ correspond to the same SO(3).

Hence SO(3) group space is RP^3: the hemisphere of S_3, i.e. $\vec{h}$ and $-\vec{h}$ are identified.



$\boxed{\text{HW: prove the } RP^3 \text{ is doubly connected!}}$