Lect 2. Basic concepts of Lie group

If a group element is denoted by continous variables, we say it's a continous group, for example, SO(3): $R(\hat{n}, \omega)$. Lie group is one type of continous group; each element is describled by a set of independent real variables. We want a one-to-one correspondence between parameter values and group elements in the regin with finite measure. [for regin with measure zero, say, the surface of the ball for SO(3) paramters, we have 2-to-one mapping]. The space for these variables is call $\boxed{\text{group space}}$, and the dimensin of the group space is the $\boxed{\text{rank}}$ of the continous group.

The product of group elements defines a motion in the group space. Assume $R \in G$, whose parameters $(r_1, r_2, \cdots r_g)$. They can be abbreviated as $R(r)$. Then $R(r)\, S(s) = T(t)$, defines

$$\boxed{t_j = f_j(r_1 \cdots r_g ; s_1 \cdots s_g) = f_j(r; s) \quad j=1, \cdots g.}$$

Then $f_j(r; s)$ for $j=1, \cdots g$ are called the composition function. If f_j's are analytic function, the group is called Lie group, which means that we can apply calculus to analyze Lie groups!

The composition functions need to satisfy

① f : group space $\times$ group space $\longrightarrow$ group space

② association : $f_j(r; f(s,t)) = f_j(f(r,s),t)$

③ The identity e : $f_j(e;r) = f_j(r;e) = r_j$.

For convenience, we often denote $e=0$.

④ The inverse of r is denoted $\bar{r}$, they satisfy

$$f_j(r;\bar{r}) = f_j(\bar{r};r) = e_j.$$

Certainly the composition functions are complicated, which are often used to prove general properties, not for concrete calculations.

§ Local properties

Consider infinitismall elements $A(\alpha)$, $B(\beta)$

① $$f_j(\alpha;\beta) = f_j(0;0) + \sum_{k=1}^{g}\left(\alpha_k \left.\frac{\partial f_j(\alpha;0)}{\partial \alpha_k}\right|_{\alpha=0} + \beta_k \left.\frac{\partial f_k(0;\beta)}{\partial \beta_k}\right|_{\beta=0}\right)$$

since $f_j(e;e) = e_j$, or $f_j(0;0) = 0$

$f_j(\alpha,e) = \alpha_j \Rightarrow \partial f_j(\alpha;0)/\partial\alpha_k = \delta_{kj}$

$\Rightarrow$ $f_j(\alpha;\beta) = \alpha_j + \beta_j \xrightarrow[\text{product between}]{} $ infinitsmal elements

$\Rightarrow$ parameters add together

② $f_j(r;\bar{r}) = e_j \Rightarrow r_j = -\bar{r}_j$.

3: Generater and their representation

Consider to use operators to represent group $\{P_G\}$,

Denote P_R is the operator corresponding to the element R. For Scalar wavefunctions, we have $P_R \psi(x) = \psi(R^{-1}x)$.

Take R as the infinismal element $A(\alpha)$, expand the expression according to α_j:

$$P_A \psi(x) = \psi(x) + \sum_j \frac{\partial \psi(A^{-1}x)}{\partial \bar{\alpha}_j} \cdot \bar{\alpha}_j \Big|_{\bar{\alpha}=0}$$

$$= \psi(x) + \sum_j \bar{\alpha}_j \sum_a \frac{\partial (A^{-1}x)_a}{\partial \bar{\alpha}_j}\Big|_{\bar{\alpha}=0} \frac{\partial \psi(A^{-1}x)}{\partial (A^{-1}x)_a}\Big|_{\bar{\alpha}=0}$$

$$\frac{\partial (A^{-1}x)_a}{\partial \bar{\alpha}_j}\Big|_{\bar{\alpha}=0} = \frac{\partial (Ax)_a}{\partial \alpha_j}\Big|_{\alpha=0}, \quad \bar{\alpha}_j = -\alpha_j, \quad \frac{\partial \psi(A^{-1}x)}{\partial (A^{-1}x)_a} = \frac{\partial}{\partial x_a}\psi(x)$$

$$\Rightarrow P_A \psi(x) = \psi(x) - i \sum_{j=1}^{g} \alpha_j \underbrace{\left(-i \sum_a \frac{\partial (Ax)_a}{\partial \alpha_j} \cdot \frac{\partial}{\partial x_a}\right)} \psi(x)$$

define differential operators

$$\boxed{\begin{aligned} &I_j = -i \sum_a \frac{\partial (Ax)_a}{\partial \alpha_j}\Big|_{\alpha=0} \frac{\partial}{\partial x_a}, \quad \text{then} \\ &P_A \psi(x) = \psi(x) - i \sum_{j=1}^{g} \alpha_j I_j \psi(x) \end{aligned}}$$

For example: For 3d rotation group

$$(Ax)_a = \sum_b \left\{ \delta_{ab} - i \sum (T_d)_{ab} \alpha_d \right\} x_b = x_a - \sum_{bd} \alpha_d \epsilon_{dab} x_b$$

$$\Rightarrow \quad I_j = -i \sum_a \frac{\partial (Ax)_a}{\partial \alpha_j} \frac{\partial}{\partial x_a} = -i \, \epsilon_{jab} x_b \frac{\partial}{\partial x_a}$$

or $\boxed{I_a = \epsilon_{abc} x_b \left(-i \frac{\partial}{\partial x_c}\right)}$ ← orbital angular momentum generator

(*) Assume m basis of wavefunction $\psi_\mu(x)$ span an invariant space of P_G, → the corresponding representation $D(G)$.

$$P_R \psi_\mu(x) = \sum_\nu \psi_\nu(x) D_{\nu\mu}(R).$$

For infinitsmal elements, $D(A) = 1 - i \sum_{j=1}^{g} \alpha_j I_j$ with $I_j = i \frac{\partial D}{\partial \alpha_j}\Big|_{\alpha}$

Then matrices I_j are called generaters of the representations $D(G)$.

Unitary representation's generaters are Hermitian matrix.

Adjonit Rep: the generaters themselves span a vector space, which can be used to form a representation, such a Rep is called adjonit Rep.

Consider $\quad RSR^{-1} = T$ and $D(R) D(S) D^{-1}(R) = D(T)$.

the group paramter satisfies $t_j = \psi_j(s_1 \cdots s_g ; r_1, \cdots r_g) = \psi_j(s; r)$

we have: take derivatives with respect to s_j

$$D(R)\,\frac{\partial D(S)}{\partial s_j}\,D(R^{-1})\Big|_{S=0} = \sum_k \frac{\partial D(T)}{\partial t_k}\,\frac{\partial \psi_k(S,r)}{\partial s_j}\Big|_{S=0}$$

$$\boxed{D(R)\,I_j\,D(R^{-1}) = \sum_k I_k\,D^{ad}_{kj}(R)}$$ where $D^{ad}_{kj}(R) = \frac{\partial \psi_k(S;r)}{\partial s_j}\Big|_{S=0}$.

· R maps to $D^{ad}(R)$ ← matrix with dimension $\# = g$.

If we take the scalar wavefunctions for Rep, the $D(R) \to P_R$, $I_j \to$ differential operator $I_j^{(0)}$, we have.

$$P_R\,I_j^{(0)}\,P_R^{-1} = \sum_k I_k^{(0)}\,D^{ad}_{kj}(R)$$

For example; in terms of angular momentum

$$\boxed{\begin{gathered} \mathcal{U}(\hat{n},\omega)\,L_j\,\mathcal{U}^{-1}(\hat{n},\omega) = \sum_k L_k\,R_{kj}(\hat{n},\omega) \\ \text{with } \mathcal{U}(\hat{n},\omega) = \exp[-i\,\vec{\omega}\cdot\vec{L}] \end{gathered}}$$

For SO(3) group, the adjoint Rep = its fundamental vector Rep.
$\Rightarrow$ angular momentum is a pseudo vector. This is not true for other SO(n).

§ Global properties

* Connectivity: If any two elements in the group space can be connected by a continous curves, then the group space is connected.

For example: SO(3) group space is connected, but O(3) is not.

O(3) has two connected componets with det $= \pm 1$, respectively.

Its component with det $= -1$, can be achieved by choosing a representi

element, say σ, and then we can derive other elements.

HW: Prove that for a disconnected Lie group, its componet with the identity element form an invariant subgroup, (normal subgroup). Hence the disconnected Lie group is completedly determined by a connect normal subgroup, and a quotiet group spanned by a set of representative elements from each component.

* Even a group space is connected, they can still be classified according to the topo structure of the lines from the identity to any element. All for simplity, just consider closed loops. We have already seen the difference between SU(2) and SO(3).

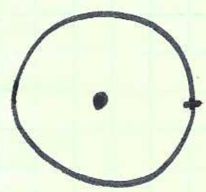
Example: ① translation group of 1D free space

$$D^k(\delta) = \exp[-ik\delta], \quad \text{— } k\text{: momentum, } \delta\text{: distance of translation}$$

This group is simply connected, All the lines in the group space (1 D real axis) can be contineously changed.

② The above system put on a ring, or, impose a period boundary condition. i.e. the translations of distance δ and $\delta+L$ are identified.

This group is isomorphic to SO(2). The loops in the group space can be classified by link number



※ If a connected Lie group is n-connected, it's isomorphic to a singly connected Lie group. This singly connected Lie group is the covering group. The representation of the covering group is actually the n-value representation.

※ SU(2) is doubly covering group of SO(3), which is the math reason of half-integer spin!