Lect 5: Character tables for finite groups

* General guidance

① $\sum_j m_j^2 = |G|$ ② $\sum_j 1 = n_c$

③ $\sum_{\alpha=1}^{n_c} n_\alpha \chi_i^*(C_\alpha)\chi_j(C_\alpha) = |G|$ ④ $\sum_j \chi_j^*[C_\alpha]\chi_j[C_\beta] = \frac{|G|}{|n_\alpha|}\delta_{\alpha\beta}$

where j is the index for irreducible Reps, n_c is # of classes,

n_α is the # of elements in class C_α, $|G|$ is the order of group.

(*) Other properties

① any group has the identity Rep.

② Any irreducible representations of the quotient group is also an irreducible Rep of the original group. — looking for invariant subgroups.

③ If G has irreducible complex representations, it's characters must have complex values. Then its complex conjugation is also an non-equivalent irreducible Rep.

④ For group G, it's 1D irreducible representation multiplying any other irreducible representation is also an irreducible representation.

⑤ The irreducible representation for the group G is also a representation of its subgroup H. But it is often reducible.

Nevertheless, the possibility of reducing is often limited.

6 If $G = H_1 \otimes H_2$, then the irreducible representations of G can be represented as $\underbrace{}_{\text{direct product of}}$ irreducible representations of H_1 and H_2.

* Cyclic group:

$$\boxed{\begin{array}{l} C_N = \{E, R, R^2, \cdots R^{N-1}\}, \quad R^N = E \\ \text{N irreducible 1D representations. The j-th representation} \\ \quad D^j(R) = e^{-i2\pi j/N} \quad \text{for } 0 \le j \le N-1 \\ \text{and hence } D^j(R^m) = e^{-i2\pi mj/N} \quad \text{for } 0 \le m \le N-1 \end{array}}$$

	E	R
A	1	1
B	1	-1

C_2

	E	R	R^2	m
A	1	1	1	0
E	1	ω	ω^2	1
E'	1	ω^2	ω	-1

C_3

$\omega = e^{-i2\pi/3}$

• discrete angular momentum

	E	R	R^2	R^3
A	1	1	1	1
B	1	-1	1	-1
E	1	$-i$	-1	i
E'	1	i	-1	$-i$

C_4

	E	R^2	R^4	R^3	R^5	R
A	1	1	1	1	1	1
B	1	1	1	-1	-1	-1
E_1	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
E_2	1	ω^2	ω	1	ω^2	ω
E_1'	1	ω^2	ω	-1	$-\omega^2$	$-\omega$
E_2'	1	ω	ω^2	1	ω	ω^2

$C_6 = C_3 \otimes C_2 = \{E, R^2, R^4\} \otimes \{E, R^3\}$

Question: How many complex Reps and real Reps ?

Answer: Consider a class. If for every element g in the class, then g^{-1} is also in this class, then it's called self-invese class.

The # of self-conjugate (real and pseudo-real) representations equals the # of self-inverse classes, and the # of complex representat equals the # of non self-inverse classes.

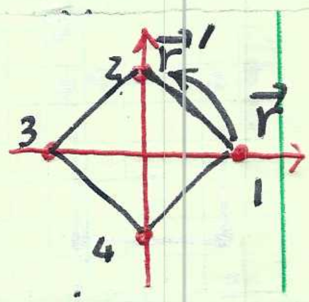
For C_4 and C_6, rotation $180°$ is a self-inverse operation also forming a class. Including the identity, they have two real representations. For odd order cyclic groups, they only have the identity Rep as a real one.

★ Application: QM — molecular orbit

① Symmetry operation on QM wave functions

Consider a symmetry operation g, it's operation on space coordinate: $\vec{r}\,' = g\vec{r}$.

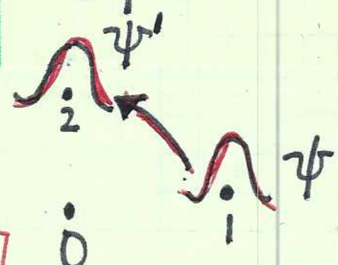
For example, g: rotation $\frac{\pi}{2}$.



How does it apply to a scalar wavefunction $\psi(\vec{r})$?

Denote the state after operation $\psi' = g\psi$. We want

$$\boxed{\psi'(\vec{r}\,') = \psi(\vec{r})}$$



$$\psi'(g\vec{r}) = \psi(\vec{r})$$

$$\Rightarrow \boxed{\psi'(\vec{r}) = \psi(g^{-1}\vec{r})}$$

in the Dirac notation $|1\rangle \Leftrightarrow \delta(r-r_1)$, $|2\rangle \Leftrightarrow \delta(r-r_2)$

$$g|1\rangle \to |2\rangle \Leftrightarrow \delta(r-r_2) = \delta(g^{-1}r - r_1) \leftarrow \text{check: if } r=r_2$$

$$g^{-1}r_2 = r_1 \ \checkmark .$$

Now suppose on each point (atom), there exist an atomic orbital $|i\rangle$, then the four orbits form a representation of C_4. under this set of basis, we have the matrices of E. R. R^2 R^3

$$E: \begin{array}{c|c} & |1\rangle\ |2\rangle\ |3\rangle\ |4\rangle \\ \begin{matrix}\langle 1| \\ \langle 2| \\ \langle 3| \\ \langle 4|\end{matrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{array} \qquad R: \begin{array}{c|c} & |1\rangle\ |2\rangle\ |3\rangle\ |4\rangle \\ \begin{matrix}\langle 1| \\ \langle 2| \\ \langle 3| \\ \langle 4|\end{matrix} & \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix} \end{array}, \qquad R^2: \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \qquad R^3 \ldots$$

hence $\boxed{\chi(E) = 4, \quad \chi(R) = \chi(R^2) = 0}$ $\swarrow \chi(R^3)$

Now we can use the character table of C_4 to decompose this representation into irreducible ones as

$$\#\text{ of } A: \ \frac{1}{4}\left[\sum_g \chi_A^*(g)\,\chi(g)\right] = \tfrac{1}{4}\cdot 4 = 1$$

$$\#\quad B, E, E', \quad = \tfrac{1}{4}\cdot 4\cdot 1 = 1$$

$$\Rightarrow \quad \boxed{A \oplus B \oplus E \oplus E'}$$

Let use form the orbits belonging to A, B, E and E' Reps.

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left[\,|1\rangle + |2\rangle + |3\rangle + |4\rangle\,\right]$$ S-wave

$\|\psi_B\rangle = \frac{1}{\sqrt{2}}[\,\|1\rangle - \|2\rangle + \|3\rangle - \|4\rangle]$	d-wave	m: 2
$\|\psi_E\rangle = \frac{1}{\sqrt{2}}[\,\|1\rangle + i\|2\rangle - \|3\rangle - i\|4\rangle]$	p-wave	1
$\|\psi_{E'}\rangle = \frac{1}{\sqrt{2}}[\,\|1\rangle - i\|2\rangle - \|3\rangle + i\|4\rangle]$	p-wave	-1

Compare with the angular momentum eigenstate $\frac{1}{\sqrt{2\pi}} e^{im\theta}$ with

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi \Longleftrightarrow |1\rangle, |2\rangle, |3\rangle, |4\rangle$, we have $m = 0, 2, \pm 1$

for $|\psi_A\rangle$, $|\psi_B\rangle$, $|\psi_{E,E'}\rangle$, respectively. We can also check their characters

For example

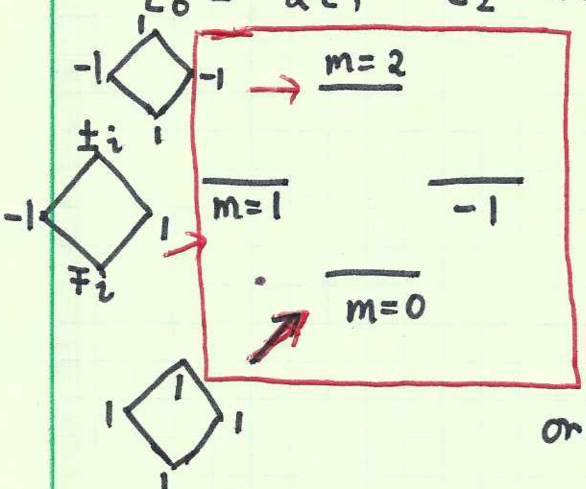
$$R|\psi_E\rangle = \frac{1}{\sqrt{2}}[\,|2\rangle + i|3\rangle - |4\rangle - i|1\rangle] = \frac{-i}{\sqrt{2}}[\,|1\rangle + i|2\rangle - |3\rangle - i|4\rangle] = -i|\psi_E\rangle.$$

$$R|\psi_{E'}\rangle = \frac{1}{\sqrt{2}}[\,|2\rangle - i|3\rangle - |4\rangle + i|1\rangle] = i|\psi_{E'}\rangle.$$

Hence, even we have not given a Hamiltnian, we have already solved the eigenfunctions!

Now consider $H = -t\sum_{\langle ij\rangle}(c_i^\dagger c_j + h.c)$, plug in the above eigen wave-functions, we have $E_m = -2t\cos\left(\frac{2\pi}{N}\cdot m\right)$ with $N = 4$

$E_0 = -2t$, $E_2 = 2t$, $E_1 = E_{-1} = 0$



(anti-bonding)

(non-bonding) $\rightarrow$ discrete angular momentum m-(mod 4): $0, \pm 1, 2$

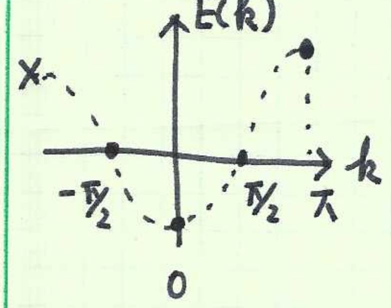
(bonding state)

or - Bloch theory: translation symmetry

$\rightarrow$ lattice momentum $k = \frac{2\pi}{N}\cdot m$

$$\psi(r) = \frac{1}{\sqrt{N}} e^{i k \cdot R_i} \qquad \bullet \ \bullet \ \bullet \ \bullet \quad N=4$$

for 1 d lattice, $\rightarrow$ translation with periodical boundary condition



$\leftarrow$ band structure

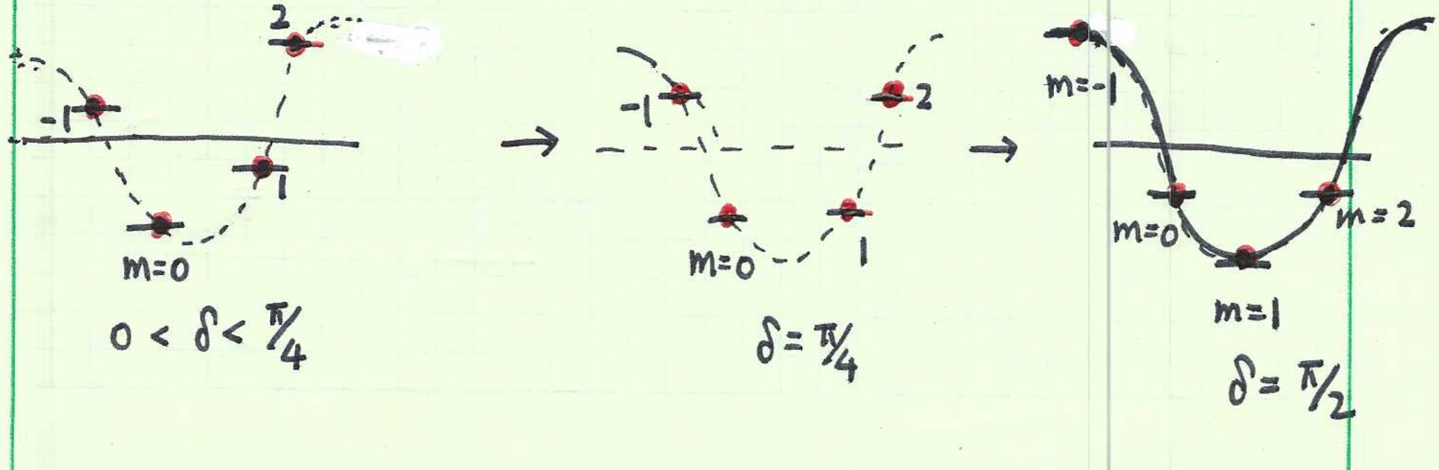
as increasing $N \to \infty$, molecule becomes crystal, and bond $\rightarrow$ band.

Question: we have seen there exist a degeneray between $|\psi_E\rangle$ and $|\psi_{E'}\rangle$. For an Abelieean group, typically we don't expect degeneray, since each representation is 1D, and different states donot transform into each other under symmetry operations. Indeed, if we put a flux ϕ into the plaqutte, it does not change the C_4 symmetry. The

$$H = -t \sum \left[C_{i+1}^{\dagger} C_i \, e^{i\delta} + h.c \right] \quad \text{with } \frac{\phi}{4} = \delta$$

then the $|\psi_{A,B,E,E'}\rangle$ remain unchanged. But their energies

$$E_m = -2t \cos\left(\frac{2\pi}{N} m - \delta\right).$$



In order to explain the degeneracy pattern at $\delta=0$, we need a higher (7)

symmetry D_4.

(8) The dihedral group

D_N group is the symmetry group of regular N-polygons. The z-axis is the N-fold axis. The C_N-group is an invariant subgroup, and quotient group is C_2. In the xy-plane, there evenly distributs N two-fold axes. D_{2N+1} and D_{2N} have different structures, and we study separately.

(1) D_{2N+1}. All the 2N+1 two-fold axes are equivalent forming one class. For the z-axis, since rotations around $\pm\hat{z}$ axis can be related by the in-plane 2-fold rotation, $R_z(\frac{\pm i}{2N+1} 2\pi)$ belong to the same class. Hence, the C_{2n+1} subgroup contains n+1 classes. In total there're n+2 classes, and then n+2 irreducible representations

$$\boxed{\sum_{j=1}^{n+2} m_j^2 = 4n+2 \,. \Rightarrow \underset{A1}{1^2} + \underset{A2}{1^2} + \underbrace{2^2 + \cdots \; 2^2}_{n} = 4n+2}$$

D_{2N+1} must have two non-equivalent 1D representation, corresponding to quotient group C_2, denoted as A1 (identity) and A2. The character table has the following structure

	E	$2C^1_{2n+1}$	$\cdots$	$(2n+1)C'_2$
A_1	1	1	1	1
A_2	1	1	1	-1
E	2	$\cdots$	$\cdots$	?

(a) look at the class of $2n+1$ two-fold axes, the characters

$$\sum_j |\chi_j(C_2')|^2 = \frac{|G|}{n_{C_2'}} = 2 .$$

Since $\chi_{A_1}(C_2') = 1$, and $\chi_{A_2}(C_2') = -1$, $\Rightarrow$ $\chi_E(C_2') = 0$ for all the other 2D representations.

(b) All the classes in D_{2n+1} are self-inverse classes, hence all representations are self-conjugate, (actually real to be checked later!). The characters should be real.

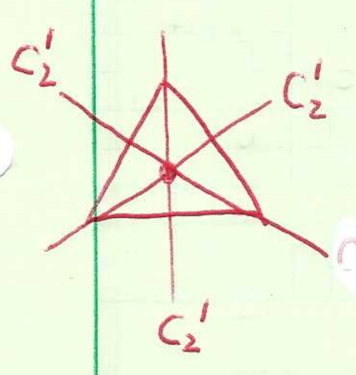
(c) Each 2D representation should be decomposed into a pair of 1D representations. Hence, it should be decomposed into a pair of complex conjugation representations. $\Rightarrow$

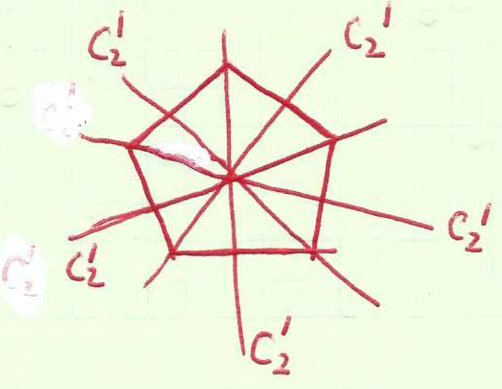
$$\chi_{E_j}(E) = 2, \quad \chi_{E_j}(C_{2n+1}^m) = 2\cos\frac{2jm\pi}{2n+1}, \quad \chi_{E_j}(C_2') = 0.$$

two examples D_3 and D_5

D_3	E	$2C_3$	$3C_2'$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

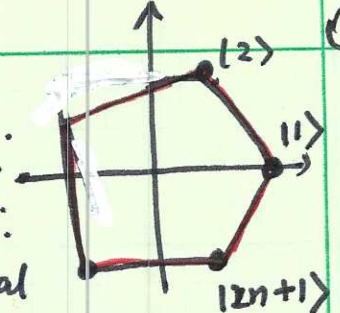
D_5	E	$2C_5^1$	$2C_5^2$	$5C_2'$
A_1	1	1	1	1
A_2	1	1	1	-1
E_1	2	p	$-p^{-1}$	0
E_2	2	$-p^{-1}$	p	0





where $p = 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}$

$p^{-1} = -2\cos\frac{4\pi}{5} = \frac{\sqrt{5}+1}{2}$



Now we use QM states to represent D_{2n+1} group.
Consider states $|1\rangle, |2\rangle, \dots |2n+1\rangle$ at sites of a regular polygon. They form a $2n+1$ dimensional representation. The rotation of C_{2n+1} operation : no-fixed sites $\Rightarrow \chi(C_{2n+1}^{l}) = 0$.
The 2-fold rotation has a fixed point, $\Rightarrow \chi(C_2') = 1$, Hence, we have

	E	$2C_{2n+1}^{1}$	$\cdots$	$2C_{2n+1}^{n}$	$\overbrace{C_2'}^{2n+1}$
χ	$2n+1$	0	0	0	1

\# of A_1 : $\frac{1}{|G|}\sum_g \chi_{A_1}^* \chi = \frac{1}{4n+2}(2n+1+2n+1) = 1$

A_2 : $\frac{1}{|G|}\sum_g \chi_{A_2}^* \chi = \frac{1}{4n+2}(2n+1-(2n+1)) = 0$

E_m : $\frac{1}{|G|}\sum_g \chi_{E_m}^* \chi = \frac{1}{4n+2}[4n+2] = 1$

Hence $\boxed{\chi = A_1 \oplus E_1 \oplus \cdots \oplus E_m}$ $\leftarrow$ $E_{1\sim m}$ are 2-dimesional reps $\Rightarrow$ 2-fold degeneracy.

$$|\psi_{A_1}\rangle = \frac{1}{\sqrt{2n+1}}\sum_{i=1}^{2n+1}|i\rangle$$

For E_1, it include the basis with (discrete) angular momentum ± 1

$$|\psi_+^{E_1}\rangle = \frac{1}{\sqrt{2n+1}}\sum_{j=1}^{2n+1} e^{-i\frac{2\pi}{2n+1}(j-1)}|j\rangle$$

$$|\psi_-^{E_1}\rangle = \frac{1}{\sqrt{2n+1}}\sum_{j=1}^{2n+1} e^{i\frac{2\pi}{2n+1}(j-1)}|j\rangle$$

under this basis, we have : the cyclic rotations C_{2n+1}^{l} is diagonal

$$\Rightarrow \begin{bmatrix} e^{-i\frac{2\pi}{2n+1}l} & 0 \\ 0 & e^{i\frac{2\pi}{2n+1}l} \end{bmatrix}$$

The 2-fold rotation: $C_2' \to$ Rotation around x-axis

$C_2'(\hat{x}): \ |1\rangle \to |1\rangle, \ |2\rangle \leftrightarrow |2n+1\rangle, \ \cdots \ |n+1\rangle \leftrightarrow |n+2\rangle$

$$\Rightarrow C_2'(\hat{x})|\psi_+^{E_1}\rangle = \frac{1}{\sqrt{2n+1}} \sum_{j=1}^{2n+1} e^{-i\frac{2\pi}{2n+1}(j-1)} |2n+3-j\rangle \leftarrow \begin{matrix} j' = 2n+3-j \\ j = 2n+3-j' \end{matrix}$$

$$= \frac{1}{\sqrt{2n+1}} \sum_{j'=1}^{2n+1} e^{-i\frac{2\pi}{2n+1}(2n+2-j')} |j'\rangle = \frac{1}{\sqrt{2n+1}} \sum_{j'=1}^{2n+1} e^{i\frac{2\pi}{2n+1}(j'-1)} |j'\rangle$$

$$= |\psi_-^{E_1}\rangle$$

Similarly $\quad C_2'(\hat{x})|\psi_{1-}^{E_1}\rangle = |\psi_+^{E_1}\rangle \quad \Rightarrow \boxed{C_2'(\hat{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

How about other C_2', — axis at angle $\frac{2\pi}{2n+1}$?

Consider the similar transformation of a rotation

$R' = R \, R_{\hat{n}}(\theta) \, R^{-1}$ — check the direction $\hat{n}' = R\hat{n}$

then $R'\hat{n}' = R \cdot R_{\hat{n}}(\theta) R^{-1} R\hat{n} = R\hat{n} = \hat{n}'$, hence the rotation axis changes to

Hence the Representation matrix for the C_2 with axis at the angle of φ is

$$\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi} & \\ & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i2\varphi} \\ e^{i2\varphi} & 0 \end{pmatrix}$$

in Sumary $\boxed{E_1\text{-rep}}$

For C_{2n+1}^{l} rotation: $\to \begin{pmatrix} e^{-i\frac{2\pi}{2n+1}l} & 0 \\ 0 & e^{i\frac{2\pi}{2n+1}l} \end{pmatrix}$

C_2 around axis at $\frac{2\pi}{2n+1} l$-angle $\to \begin{pmatrix} 0 & e^{-i\frac{4\pi}{2n+1}l} \\ e^{i\frac{4\pi}{2n+1}l} & 0 \end{pmatrix}$

In many situations, we want to real basis (chemists like this set)

$$|\psi_x^{E_1}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_+^{E_1}\rangle + |\psi_-^{E_1}\rangle\right) = \sqrt{\frac{2}{2n+1}}\sum_{j=1}^{2n+1} \cos\frac{2\pi}{2n+1}(j-1)\,|j\rangle \leftarrow P_x$$

$$|\psi_y^{E_1}\rangle = \frac{i}{\sqrt{2}}\left(|\psi_+^{E_1}\rangle - |\psi_-^{E_1}\rangle\right) = \sqrt{\frac{2}{2n+1}}\sum_{j=1}^{2n+1} \sin\frac{2\pi}{2n+1}(j-1)\,|j\rangle \leftarrow P_y$$

Then
$$\left(|\psi_x^{E_1}\rangle, |\psi_y^{E_1}\rangle\right) = \left(|\psi_+^{E_1}\rangle, |\psi_-^{E_1}\rangle\right)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \leftarrow \text{define as } \mathcal{U}$$

$\Rightarrow$ The representation matrix

$$\begin{pmatrix} \langle\psi_E^x| \\ \langle\psi_E^y| \end{pmatrix} O \left(|\psi_E^x\rangle, |\psi_E^y\rangle\right) = \mathcal{U}^{+} \underbrace{\begin{pmatrix} \langle\psi_+^E| \\ \langle\psi_-^E| \end{pmatrix} O \left(|\psi_+^{E_1}\rangle, |\psi_-^{E_1}\rangle\right)}_{\text{Representation matrix in the previous basis}} \mathcal{U}$$

$\Rightarrow$ The $C_{2n+1}^{l} \rightarrow$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} e^{-i\frac{2\pi}{2n+1}l} & 0 \\ 0 & e^{i\frac{2\pi}{2n+1}l} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{2n+1}l & \sin\frac{2\pi}{2n+1}l \\ -\sin\frac{2\pi}{2n+1}l & \cos\frac{2\pi}{2n+1}l \end{pmatrix}$$

C_2:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & e^{-i\frac{4\pi}{2n+1}l} \\ e^{i\frac{4\pi}{2n+1}l} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\frac{4\pi}{2n+1}l & -\sin\frac{4\pi}{2n+1}l \\ -\sin\frac{4\pi}{2n+1}l & -\cos\frac{4\pi}{2n+1}l \end{pmatrix}$$

Clearly, this is a real representation.

For other representations E_m: under the angular momentum eigen-basis $\pm m$

$$|\psi^{E_m}_{\pm}\rangle = \frac{1}{\sqrt{2n+1}} \sum_{j=1}^{2n+1} e^{\mp i \frac{2\pi}{2n+1} m(j-1)} |j\rangle$$

The cyclic rotation C_{2n+1}^{l} : $\begin{bmatrix} e^{-i\frac{2\pi}{2n+1} ml} & 0 \\ 0 & e^{i\frac{2\pi}{2n+1} ml} \end{bmatrix}$

and the C_2 rotation around

the axis at the angle $\frac{2\pi}{2n+1} l$ $\quad C_2^{l}$: $\begin{bmatrix} 0 & e^{-i\frac{4\pi}{2n+1} ml} \\ e^{i\frac{4\pi}{2n+1} ml} & 0 \end{bmatrix}$

Again this Rep is actually real: under the basis of

$$(|\psi^{E_m}_{\cos}\rangle, |\psi^{E_m}_{\sin}\rangle) = (|\psi^{E_m}_{+}\rangle \; |\psi^{E_m}_{-}\rangle) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

we have : C_{2n+1}^{l} : $\begin{bmatrix} \cos\frac{2\pi}{2n+1} ml & \sin\frac{2\pi}{2n+1} ml \\ -\sin\frac{2\pi}{2n+1} ml & \cos\frac{2\pi}{2n+1} ml \end{bmatrix}$

the lth-C_2^{l} : $\begin{bmatrix} \cos\frac{4\pi}{2n+1} ml & -\sin\frac{4\pi}{2n+1} ml \\ -\sin\frac{4\pi}{2n+1} ml & -\cos\frac{4\pi}{2n+1} ml \end{bmatrix}$

In general

	E	$2C_{2n+1}^{1}$	$2C_{2n+1}^{2}$	$\cdots$	$2C_{2n+1}^{l}$	$\cdots$	$2C_{2n+1}^{n}$	$(2n+1)C_2$
A_1	1	1	1	1	1	$\cdots$	1	1
A_2	1	1	1	1	1	$\cdots$	1	-1
E_1	2	$2\cos\frac{\pi}{2n+1}$	$2\cos\frac{2\pi}{2n+1}$	$\cdots$	$2\cos\frac{2\pi l}{2n+1}$	$\cdots$	$2\cos\frac{2\pi n}{2n+1}$	0
$\vdots$								
E_m	2	$2\cos\frac{m\pi}{2n+1}$	$2\cos\frac{2m\pi}{2n+1}$	$\cdots$	$2\cos\frac{ml}{2n+1}\cdot 2\pi$		$2\cos\frac{2\pi mn}{2n+1}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
E_n	2	$2\cos\frac{n\pi}{2n+1}$	$2\cos\frac{2n\pi}{2n+1}$	$\cdots$	$2\cos\frac{nl}{2n+1}2\pi$		$2\cos\frac{n^2}{2n+1}\cdot 2\pi$	0

The analysis for the non-abealian group D_{2n} can be done similarly. which will be left for homework.

Below let me show the Rep of D_2 group. $= \{E,\ R_x(\pi),\ R_y(\pi),\ R_z(\pi)\}$.

It's an Abelian group, hence it has 4-representations. It has 3 normal subgroups $\{E, R_z(\pi)\}$, $\{E, R_x(\pi)\}$, and $\{E, R_y(\pi)\}$. For each of them, their quotient group is simply Z_2, and can generate a representations for D_2.

For example:

$$\{E, R_z(\pi)\} \longrightarrow E$$
$$R_x\{E, R_z(\pi)\} \longrightarrow R_x$$

This gives rise to two representations mapping $\{E, R_z(\pi)\}$ to 1. They are called A_1 and A_2.

A_1: if $R_x \to 1$ $\quad$ A_2: if $R_x \to -1$.

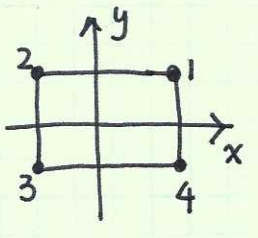
$$\{E, R_x(\pi)\} \longrightarrow E$$
$$R_y\{E, R_x(\pi)\} \longrightarrow R_y$$

This will generate another new 1d Representation we call it as B. If we use $\{E, R_y(\pi)\}$ as a normal subgroup, we get another one. These two are called B_1, B_2.

	E	$R_z(\pi)$	$R_x(\pi)$	$R_y(\pi)$
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1

These 4-1D representations remain for other D_{2n} groups.

If we use a rectangular geometry



$$\to |\psi_{A_1}\rangle = \frac{1}{2}(|1\rangle + |2\rangle + |3\rangle + |4\rangle)$$
$$|\psi_{A_2}\rangle = \frac{1}{2}(|1\rangle - |2\rangle + |3\rangle - |4\rangle)$$
$$|\psi_{B_1}\rangle = \frac{1}{2}(|1\rangle - |2\rangle - |3\rangle + |4\rangle)$$
$$|\psi_{B_2}\rangle = \frac{1}{2}(|1\rangle + |2\rangle - |3\rangle - |4\rangle)$$

If n is even, we have integer angular momentum

$$\boxed{\begin{array}{l} j_z = 0, \pm 1, \cdots \pm(\frac{n}{2}-1), \frac{n}{2} \\ \quad \pm 1/2, \pm 3/2, \cdots \pm \frac{n-1}{2} \\ \chi_{j_z}(R^l) = e^{-i j_z l \theta} \quad \text{with } \theta = \frac{2\pi}{n} \end{array}}$$

Again only $j_z = 0, \frac{n}{2}$, the representations are real, otherwise they are complex. $\rightarrow \chi_{j_z = \frac{n}{2}}(R^l) = (-)^l$.

Double group of dihedral groups

Let's begin with D_2, whose double group is the Quarternion group $Q \simeq D_2^D$.

The E and $\bar{E}$ are two classes, since they commute with all other elements. $R_x(\pi)$ and $\bar{R}_x(\pi) = R_x(3\pi)$ are also in the same class, since $\bar{R}_x(\pi) = R_x(-\pi) = R_{-\hat{x}}(\pi)$ and the $\hat{x}$ is a bilateral axis. So do $R_y(\pi)$ and $\bar{R}_y(\pi)$, $R_z(\pi)$ and $\bar{R}_z(\pi)$. Hence there are 5 classes. Q or D_2^D has a normal subgroup $\{E, \bar{E}\}$, thus the quotient group is D_2, hence all the previous four 1D representations remain ~~an~~ representation of D_2. According to

$$\underbrace{1^2 + 1^2 + 1^2 + 1^2}_{\substack{D_2 \\ \{E,\bar{E}\} \to 1}} + \underbrace{2^2}_{\text{spinor Rep.}} = 8.$$

we have a new spin-1/2 representation

(*) Now we conside double group — discrete subgroup of $SU(2)$

For $SU(2)$ group, rotation $360°$ is different from $0°$. We denote this element as $\bar{E}$. $\bar{E}$ also commutes with all other elements, The double group has twice number of elements compared with the original group.

Double group of the cyclic group — Z_n

$$\{\underset{E}{R_z(0)},\ R_z(\tfrac{2\pi}{N}),\ \cdots\ \underset{\bar{E}}{R_z(2\pi)},\ \cdot\ \underset{\bar{E}R_z(\frac{2\pi}{N})}{R_z((1+\tfrac{1}{N})2\pi)}\ \cdots\ R_z[(2-\tfrac{1}{N})2\pi]\}$$

This remains an Abelian group, isomorphic to Z_{2n}, hence, it's not essentially a new group. Nevertheless, the physically intepretation of Reps are different.

- If n is odd, we have integer angular momentum Rep with $j_z = 0, \pm 1, \cdots \pm (n-1)/2$. — These are representations of Z_n.

Now j_z can also take half integer angular momentum

$$j_z = \pm 1/2, \pm 3/2, \cdots \pm \frac{n-2}{2}, \frac{n}{2} \leftarrow \frac{n}{2} \equiv -\frac{n}{2} \pmod n$$

which are the same Reps.

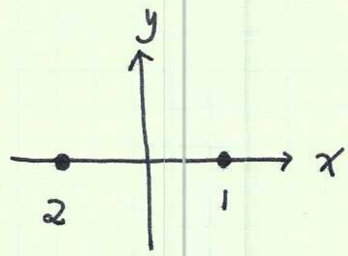
j_z	E	R	R^2	$\cdots$	R^{n-1}	$\bar{E}$	$\bar{E}R$	$\cdots$	$\bar{E}R^{n-1}$
0	1	1	1	1	1	1	1	1	1
$\pm m$	1	$e^{\mp im\theta}$	$e^{\mp 2im\theta}$	$\cdots$	$e^{\mp i(n-1)\theta}$	1	$e^{\mp im\theta}$	$\cdots$	$e^{\mp i(n-1)\theta}$
$\vdots$									
$\pm 1/2$	1	$e^{\mp i\frac{\theta}{2}}$	$\cdots$		$e^{\mp i\frac{n-1}{2}\theta}$	-1	$-e^{\mp i\frac{\theta}{2}}$	$\cdots$	$-e^{\mp i\frac{n-1}{2}\theta}$
$\vdots$	$\vdots$	$\vdots$							
$\frac{n}{2}$	1	-1	1	$\cdots$	1	-1	1	$\cdots$	-1

$$\theta = \frac{2\pi}{n}$$

↳ c.f. the character table of Z_{2n}.

Then we have

	E	$\bar{E}$	$R_z\ \bar{R}_z$	$R_x\ \bar{R}_x$	$R_y\ \bar{R}_y$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	1	-1	1	-1
B_2	1	1	-1	-1	1
$E_{1/2}$	2	-2	0	0	0



$|1\uparrow\rangle, |1\downarrow\rangle$

$|2\uparrow\rangle, |2\downarrow\rangle$

Let's consider just 2 points, but now with spin. They form a 4-d representation. Under $R_z, \bar{R}_z\ R_x\ \bar{R}_x\ R_y\ \bar{R}_y$, there're no diagonal terms, their characters are simply 0. Then $\chi = (4, 4, 0, 0, 0)$, hence it contains two $E_{1/2}$, i.e. $E_{1/2} \oplus E_{1/2}$.

We can choose $|\psi_{E_{1/2}\uparrow}\rangle = \frac{1}{\sqrt{2}}\left[|1\uparrow\rangle + |2\uparrow\rangle\right]$, $|\psi_{E_{1/2}\downarrow}\rangle = \frac{1}{\sqrt{2}}\left[|1\downarrow\rangle + |2\downarrow\rangle\right]$

note $e^{-i\frac{\sigma_z}{2}\pi} = \begin{pmatrix} -i & \\ & i \end{pmatrix}$, $e^{-i\frac{\sigma_x}{2}\pi} = -i\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $e^{-i\frac{\sigma_y}{2}\pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Hence: $E\,(\bar{E})$: $\begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}$, $R_z(\bar{R}_z)$: $\begin{pmatrix} \mp i & \\ & \pm i \end{pmatrix}$,

$R_x(\bar{R}_x)$: $\begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix}$ $R_y(\bar{R}_y)$: $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$

$$\frac{1}{|G|}\sum_g \chi(g^2) = \frac{1}{8}\left(\chi(E) + \chi(E) + \chi(\bar{E}) + \chi(\bar{E}) + \cdots\right) = \frac{1}{8}(-4\times 2) = -1$$

$\boxed{\text{Hence it's pseudo-real!}}$ no basis for a purely real representation.

If we use $|\psi'_{\uparrow}\rangle = \frac{1}{\sqrt{2}}\left[|1\uparrow\rangle - |2\uparrow\rangle\right]$, $|\psi'_{\downarrow}\rangle = \frac{1}{\sqrt{2}}\left[|1\downarrow\rangle - |2\downarrow\rangle\right]$

then please check that, we will have

$$E\ (\bar{E}): \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \quad R_z(R_{\bar{z}}) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \quad R_x(R_{\bar{x}}) = \begin{pmatrix} 0 & \mp i \\ \mp i, & 0 \end{pmatrix}$$

$$R_y(\bar{R}_y): \begin{pmatrix} 0 & \mp 1 \\ \mp 1 & 0 \end{pmatrix}.$$

These two representations are equivalent. Can you find the matrix to connect them?