Lect 16 SO(2n+1) and SO(2n), spinors

⊛ SO(2n+1) : simple roots and fundamental weights.

L_{ab} for SO(2n+1)'s vector representation with

$$\boxed{(L_{ab})_{ij} = -i(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}).}$$

They satisfy $\boxed{[L_{ab}, L_{cd}] = i[\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}]}$

We can take the cartan subalgebra $\boxed{H_j = L_{2j-1,2j} \text{ for } j=1,2,\cdots n.}$

The roots are

$$E_{\pm e_j} = \frac{1}{\sqrt{2}}[L_{2j-1,2n+1} \pm i\, L_{2j,2n+1}] \qquad (j=1,\cdots n)$$

$$E_{e_j \pm e_k} = \frac{1}{2}[L_{2j-1,2k-1} + i\,L_{2j,2k-1} \pm i[L_{2j-1,2k} + i\,L_{2j,2k}]]$$

$$E_{-(e_j \pm e_k)} = \frac{1}{2}[L_{2j-1,2k-1} - i\,L_{2j,2k-1} \mp i[L_{2j-1,2k} - i\,L_{2j,2k}]]$$

$$(1 \le j < k \le n)$$

total # of roots $\left(n + \frac{n(n-1)}{2} \times 2\right) \times 2$

$$= 2n^2$$

Check $[H_i, E_{e_j}] = \frac{1}{\sqrt{2}}\left([L_{2i-1,2i}, L_{2j-1,2n+1}] + i[L_{2i-1,2i}, L_{2j,2n+1}]\right)$

$$= \frac{i}{\sqrt{2}}\delta_{ij}[L_{2i,2n+1} - i\,L_{2i-1,2n+1}] =$$

$$= \frac{1}{\sqrt{2}}\delta_{ij}[L_{2i-1,2n+1} + i\,L_{2i,2n+1}] = \delta_{ij}E_{e_j}$$

$$\Rightarrow \boxed{[\vec{H}, E_{e_j}] = \vec{e}_j E_{e_j}}$$

$$[H_i,\ L_{2j-1,2k-1} + iL_{2j,2k-1}] = [L_{2i-1,2i},\ L_{2j-1,2k-1} + iL_{2j,2k-1}]$$

$$= i[\delta_{ij}\, L_{2i,2k-1} - \underbrace{L_{2i,2j-1})}_{\delta_{ik}} + i\,[-i\,\delta_{ij}\, L_{2i-1,2k-1} - i\delta_{ik}\, L_{2i,2j}]$$

$$= \delta_{ij}\,[L_{2i-1,2k-1} + iL_{2i,2k-1}] + \delta_{ik}\,[L_{2i,2j} - i\,L_{2i,2j-1}]$$

$$[H_i,\ L_{2j-1,2k} + iL_{2j,2k}] = [L_{2i-1,2i},\ L_{2j-1,2k} + iL_{2j,2k}]$$

$$= i[\delta_{ij}\, L_{2i,2k} + \delta_{ik}\, L_{2i-1,2j-1}] + i\,[\delta_{ij}\,(-i)\, L_{2i-1,2k} + \delta_{ik}(i)\, L_{2i-1,2j}]$$

$$= \delta_{ij}\,[L_{2i-1,2k} + iL_{2i,2k}] + \delta_{ik}\,[-L_{2i-1,2j} + i\,L_{2i-1,2j-1}]$$

$\Rightarrow$

$$[H_i,\ E_{e_j \pm e_k}] = \frac{\delta_{ij}}{2}\,[L_{2j-1,2k-1} + iL_{2j,2k-1} \pm i\,(L_{2j-1,2k} + iL_{2j,2k})]$$

$$\pm \frac{\delta_{ik}}{2}\,[L_{2j-1,2k-1} + i\,L_{2j,2k-1} \pm i(L_{2j-1,2k} + i\,L_{2j,2k})]$$

$$= [\delta_{ij} \pm \delta_{ik}]\,(E_{e_j \pm e_k})$$

or $\boxed{[\vec{H},\ E_{e_j \pm e_k}] = (\vec{e}_j \pm \vec{e}_k)\, E_{e_j \pm e_k}}$

Hence, we can take the simple roots as

$$\boxed{\begin{cases} \vec{\alpha}_j = \vec{e}_j - \vec{e}_{j+1} & \text{for } j = 1, 2, \cdots n-1 \\ \vec{\alpha}_n = \vec{e}_n & \end{cases}}$$

$$\frac{(\alpha_i \cdot \alpha_j)}{\sqrt{(\alpha_i \alpha_i)(\alpha_j \alpha_j)}} = \frac{-1}{\sqrt{2}\cdot\sqrt{2}} \quad 1 \le i < j \le n-1$$

$\Rightarrow 120°$

SO(2n+1)

$$\frac{(\alpha_{n-1} \cdot \alpha_n)}{\quad} = \frac{-1}{\sqrt{2}} \Rightarrow 135°$$

The fundmental weight μ^i, satisfyy

$\mu_1 = (1, 0 \cdots 0)$
$\mu_2 = (1, 1, \cdots 0)$

$$\boxed{\frac{2(\alpha_i, \mu_j)}{(\alpha_i, \alpha_i)} = \delta_{ij} \quad \Rightarrow \begin{cases} \vec{\mu}_j = \sum_{k=1}^{j} \vec{e}_k & \text{for } j = 1, \cdots n-1 \\ \vec{\mu}_n = \frac{1}{2} \sum_{k=1}^{n} e_k \end{cases}}$$

$\vec{\mu}_n$: fundamental spinor

Test: For $j, j' = 1, \cdots n-1$, $\vec{\alpha}_j = \vec{e}_j - \vec{e}_{j+1}$ and $\vec{\mu}_{j'} = \sum_{k=1}^{j'} \vec{e}_k$

①

$$\vec{\alpha}_j \cdot \vec{\mu}_{j'} = \sum_{k=1}^{j'} \vec{e}_j \cdot \vec{e}_k - \sum_{k=1}^{j'} \vec{e}_{j+1} \cdot \vec{e}_k = \theta(j \le j') - \theta(j+1 \le j')$$

$$= \begin{cases} 1 - 1 = 0, & \text{if } j < j' \\ 1 - 0 = 1 & \text{if } j = j' \\ 0 - 0 & \text{if } j > j' \end{cases} \quad \left. \begin{array}{l} \Rightarrow (\vec{\alpha}_j \cdot \vec{\mu}_{j'}) = \delta_{jj'} \\ \text{and } (\alpha_j, \alpha_j) = 2 \end{array} \right\} \Rightarrow \frac{2(\alpha_j, \mu_{j'})}{(\alpha_j, \alpha_j)} = \delta_{jj'}$$

② for $i = n$ $\Rightarrow$

$$\vec{\alpha}_n \cdot \vec{\mu}_j = \vec{e}_n \cdot \begin{cases} \sum_{k=1}^{j} \vec{e}_k \\ \frac{1}{2} \sum_{k=1}^{n} e_k \end{cases} = \begin{cases} 0 & j < n \\ \frac{1}{2} & j = n \end{cases}$$

and $(\alpha_n, \alpha_n) = 1$

$$\Rightarrow \frac{2(\alpha_n, \mu_j)}{(\alpha_n, \alpha_n)} = 1$$

③ for $j = n$, $i < n$ $\Rightarrow$

$$\frac{2(\alpha_i, \mu_n)}{} = (e_i - e_{i+1}, \sum_{k=1}^{n} e_k)$$

$$= 1 - 1 = 0$$

The last weight $\vec{\mu}_n$ is different. The corresponding ~~states~~ $|\mu_n\rangle$ carries eigenvalues of H_i of $\frac{1}{2}$, hence, it's a spinor weight. According to Weyl reflection,

$$\boxed{\begin{array}{ll}
\mu_n = \tfrac{1}{2}(e_1 + \cdots\ e_n) & \\
\swarrow e_1 : q-p = \dfrac{2(\mu_n\cdot e_1)}{(e_1\cdot e_1)} = 2 & \searrow e_n \\
\tfrac{1}{2}(-e_1+e_2 \cdots\ e_n) & \tfrac{1}{2}(e_1 + \cdots - e_n) \\
\quad \swarrow e_2 \quad \cdots \quad \searrow e_n & \quad \swarrow \quad \searrow \\
\tfrac{1}{2}(-e_1-e_2+\cdots\ e_n) \quad \tfrac{1}{2}(-e_1+e_2\cdots\ e_n) & \quad \cdots\cdots
\end{array}}$$

It's easy to see that by applying the weyle reflection, along each direction of $\hat{e}_i$, i.e. flip the sign of the component along $\hat{e}_i$, we can arrive all the weights $\boxed{\mu = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_n)\text{. Such a representation's dimension is } 2^n.}$ All the weights are equivalent to the highest weight μ_n, since all of them can be related by Weyl reflection.

We can also use the previous formulae: we have the following positive roots:

$$\begin{cases} \vec{e}_1, \cdots\ \vec{e}_n \\ \vec{e}_j \pm \vec{e}_k, \quad 1 \le j < k \le n \end{cases} \qquad n + 2\times\frac{n(n-1)}{2} = n^2$$

$$\Rightarrow \rho = \frac{1}{2}\sum_{\alpha\in\Delta}\vec{\alpha} = \frac{(2n-1)}{2}\hat{e}_1 + \frac{(2n-3)}{2}\hat{e}_2 + \cdots \frac{1}{2}\hat{e}_n$$

$$\cancel{\rho\cdot e_j = 2n-(2j+1)} \Rightarrow \boxed{\begin{array}{l}\text{Casimir of SO(2n+1) fundamental spinor}\quad C_2 = \frac{1}{4}\cdot n + \frac{2n-1}{2} + \cdots \frac{1}{2} \\ \qquad = \frac{1}{4}n + \frac{n^2}{2} = \frac{n}{2}\left(n+\frac{1}{2}\right)\end{array}}$$

① The fundamental vector representation is denoted by the fundemtal weight $\vec{\mu}_1 = \hat{e}_1$, then we can imagine that by applying other roots, we end up with $\boxed{\pm \hat{e}_1, \pm \hat{e}_2, \cdots \pm \hat{e}_n \text{ and the zero weight}}$ at the origin. This is the $(2n+1)$ dimensional Rep. Then its casimir

$$\boxed{C_2 = \vec{\mu}_1 \cdot (\vec{\mu}_1 + 2\rho) = 1 + (2n-1) = 2n \text{ for fundamatal vector.}}$$

② the $\vec{\mu}_2 = \hat{e}_1 + \hat{e}_2$ corresponds to the rank-2 antisymmetric tensor

$$\left\{ \begin{array}{ll} \pm \hat{e}_i \pm \hat{e}_j & \text{with } 1 \le i < j \le n \\ \pm \hat{e}_i & i = 1, \cdots n \\ n\text{-fold zero weight} & \end{array} \right. \Rightarrow \begin{aligned} & 4 \cdot \frac{n(n-1)}{2} + 2n + n \\ &= 2n^2 - 2n + 2n + n \\ &= 2n(2n+1)/2 \end{aligned}$$

it's Casimir $C_2 = (\hat{e}_1 + \hat{e}_2)(\hat{e}_1 + \hat{e}_2) + (\hat{e}_1 + \hat{e}_2) \cdot [(2n-1)\hat{e}_1 + (2n-3)\hat{e}_2 + \cdots]$

$$= 2 + (2n-1) + (2n-3) = 4n-2$$

$SO(2n+1)$ - adjoint Rep. ~~or~~ rank-2 antisymmetric tensor

③ $\vec{\mu}_r = \hat{e}_1 + \hat{e}_2 + \cdots \hat{e}_r$ — rank-r antisymmetric tensor

The basis can be taken as $x_{i_1} \wedge x_{i_2} \wedge \cdots x_{i_r}$, $i_1 \cdots i_r \in (1, 2, \cdots 2n+1)$

$\Rightarrow$ dimension $\binom{2n+1}{r}$, ant its Casimir

$$C_2 = \vec{\mu}_r \cdot \vec{\mu}_r + \vec{\mu}_i \cdot [(2n-1)\hat{e}_1 + (2n-3)\hat{e}_2 + \cdots \hat{e}_n]$$

$$= r + (2n-1) + \cdots (2n - (2r-1)) = r + (2n-r) \cdot r$$

⊛ Reality of SO(2n+1) spiner

According to the weights $\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \cdots \pm \frac{1}{2}e_n$, they are inversion symmetric, hence this representation is real or pseudo-real. Let's construct the rank-n Clifford algebra, which generates the 2^n-dim spiner Rep.

① rank-1 : $\Gamma_1^{(n=1)} = \sigma_1$, $\Gamma_2^{(n=1)} = \sigma_2$ $\quad \Gamma_3^{(n=1)} = \sigma_3$

$$\Gamma_1 \Gamma_2 \Gamma_3 = \sigma_1 \sigma_2 \sigma_3 = i$$

② rank-2 $\quad \Gamma_1^{(n=2)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\Gamma_{2,3,4}^{(n=2)} = \begin{pmatrix} & -i\,\Gamma_{1,2,3}^{(n=1)} \\ i\,\Gamma_{1,2,3}^{(n=1)} & \end{pmatrix}$ $\quad \Gamma_5^{(n=2)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$\Gamma_1 \Gamma_2 \cdots \Gamma_5 = \begin{pmatrix} i\Gamma_1\Gamma_2\Gamma_3 & \\ & i\Gamma_1\Gamma_2\Gamma_3 \end{pmatrix} = i^2$$

③ assume we have already rank-k Γ-matrices ← $2^k \times 2^k$ dim

$$\Gamma_1^{(k)} \cdots \Gamma_{2k+1}^{(k)} = i^k, \quad \text{then}$$

$$\Gamma_1^{(k+1)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \Gamma_{2\sim 2k+2}^{(k+1)} = \begin{pmatrix} 0 & -i\,\Gamma_i^{(k)} \\ i\,\Gamma_i^{(k)} & 0 \end{pmatrix} \quad \Gamma_{2k+3}^{(k+1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\Rightarrow \Gamma_1^{(k+1)} \cdots \Gamma_{2k+3}^{(k+1)} = i^{k+1}$$

$\Rightarrow$ We can define $\Gamma_{ab}^{(k)} = -i\,\Gamma_a^{(k)}\Gamma_b^{(k)} \quad (1 \le a < b \le 2k+1)$

and the generaters in the fundemental spiner Rep is

$$L_{ab} = \frac{1}{2}\Gamma_{ab}^{(k)}$$

Define charge conjugation matrix R

$$\boxed{R\,\Gamma_{ab}R^{-1} = -\Gamma_{ab}^{*} = -\Gamma_{ab}^{T}}$$

① R can be determined up to a factor

Proof: if there's another matrix Q satisfying $Q\Gamma_{ab}Q^{-1} = -\Gamma_{ab}^{*}$

$\Rightarrow \Gamma_{ab} = -R^{-1}\Gamma_{ab}^{*}R = R^{-1}Q\,\Gamma_{ab}\,Q^{-1}R$, hence $[\Gamma_{ab}, R^{-1}Q] = 0$

for all Γ_{ab}. According to Schur's Lemma, $\Rightarrow R^{-1}Q = \lambda I$ or $R = \lambda^{-1}Q$.

② R must be either symmetric, or anti-symmetric. Since Γ_{ab} are Hermitian:

$$\begin{cases} R\,\Gamma_{ab}R^{-1} = -\Gamma_{ab}^{T} \\ (R^{-1})^{T}\Gamma_{ab}^{T}R^{T} = -(\Gamma_{ab}^{*})^{T} = -\Gamma_{ab} \end{cases}$$

$$\Rightarrow R^{T}\Gamma_{ab}(R^{-1})^{T} = -\Gamma_{ab}^{T}$$

$$\Rightarrow R^{T} = \lambda R \Rightarrow R = (R^{T})^{T} = \lambda R^{T} = \lambda^{2}R \Rightarrow \lambda = \pm 1$$

hence R can only be either symmetric or anti-symmetric.

③ If R is symmetric, there exist a transformation

$\mathcal{U}^{\dagger}\Gamma_{ab}\mathcal{U} = \Gamma_{ab}'$ such that all Γ_{ab}''s are purely imaginary, such that $\mathcal{U} = \exp[i\Gamma_{ab}]$ are real.

If R is anti-symmetric, then $\mathcal{U}$ and $\mathcal{U}^{*}$ are equivalent

$R\,\mathcal{U}R^{-1} = \mathcal{U}^{*}$. This is called pseudo-real.

$\boxed{\text{How to Choose } R?}$

1. $n=1$: Choose the purely imagniary $R = i\sigma_2$ ← an "i" is added to make R real. Then $R\,\Gamma_a\,R^{-1} = \sigma_2\,\Gamma_a\,\sigma_2 = \Gamma_a^*$

$$R\,i\Gamma_a\Gamma_b\,R^{-1} = i\,R\Gamma_a R^{-1}\,R\Gamma_b R^{-1} = i(-)^2(\Gamma_a\Gamma_b)^* = -(i\Gamma_a\Gamma_b)^*$$

$$\Rightarrow R\,\Gamma_{ab}\,R^{-1} = -\Gamma_{ab}^* \quad \text{and} \quad R^T = -R$$

2. $n=2$: Choose the 2 purely imagniary Γ-matrices and then make a product, then $R = \Gamma_2\Gamma_4 \Rightarrow R\,\Gamma_a\,R^{-1} = \Gamma_a^*$ (real; commute / imaginary: anti-commute)

$$\Rightarrow R(i\Gamma_a\Gamma_b)R^{-1} = -(i\Gamma_a\Gamma_b)^* \qquad R^T = \Gamma_4^T\Gamma_2^T = \Gamma_4\Gamma_2 = -R$$

3. $n=3$: $R = i\Gamma_{i_1}\Gamma_{i_2}\Gamma_{i_3}$ where Γ_{i_1,i_2,i_3} represents three purely imaginary Γ-matrices

$$R\,\Gamma_a\,R^{-1} = -\Gamma_a^*$$

$$\Rightarrow R\,\Gamma_{ab}\,R^{-1} = -\Gamma_{ab}^*, \qquad \boxed{R^T = \underset{\text{real}}{i}\,\Gamma_{i_3}^T\Gamma_{i_2}^T\Gamma_{i_1}^T = -i\,\Gamma_{i_3}\Gamma_{i_2}\Gamma_{i_1} = R}$$

$\Rightarrow$

∴ Generally speaking at level n, there're n-purely imaginary Γ-matrices

$$R = \Gamma_{i_1}\cdots\Gamma_{i_n}, \quad \text{which} \rightarrow R\,\Gamma_a\,R^{-1} = (-)^n\,\Gamma_a^*$$

and $R\,\Gamma_{ab}\,R^{-1} = -\Gamma_{ab}^*$.

Then

$$R^T = (-)^n\,\Gamma_{i_n}\cdots\Gamma_{i_1} = (-)^n(-)^{(n-1)+\cdots 1}\,\Gamma_{i_1}\cdots\Gamma_{i_n}$$

$$= R(-)^{\frac{n(n+1)}{2}} = -R \quad \text{if } n = 1,2,\cdots 4k+1, 4k+2$$

$$+R \quad \text{if } n = 3,4,\cdots 4k-1, 4k$$

$\left.\begin{array}{l} SO(8k+3) \\ SO(8k+5) \end{array}\right\}$ pseudo real

$\left.\begin{array}{l} SO(8k-1) \\ SO(8k+1) \end{array}\right\}$ real

(*) SO(2n) spinor

We only need to keep those roots $E_{\pm e_j \pm e_k}$ that only involve rotations ~~in~~ in the space ~~of~~ spanned 1-2n axes. And the Cartan subalgebra remain $H_j = L_{2j-1,2j}$. for $j=1,2,\cdots n$.

$$E_{e_j \pm e_k} = \frac{1}{2}\left[L_{2j-1,2k-1} + iL_{2j,2k-1} \pm i\left(L_{2j-1,2k} + iL_{2j,2k}\right)\right]$$

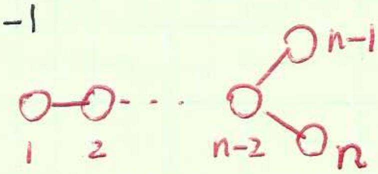
$$E_{-(e_j \pm e_k)} = E^{\dagger}_{e_j \pm e_k} \qquad (1 \le j < k \le n)$$

$$\left.\begin{array}{l} \#\text{ of roots. } \frac{n(n-1)}{2} \times 2 \times 2 = 2n^2 - 2n \\ \#\text{ of Cartan subalgebra } = n \end{array}\right\} \Rightarrow 2n^2 - n = \frac{2n(2n-1)}{2}$$

$$[\vec{H}, E_{\hat{e}_j \pm \hat{e}_k}] = (\hat{e}_j \pm \hat{e}_k) E_{\hat{e}_j + \hat{e}_k}$$

We take the simple roots.

$$\begin{cases} \vec{\alpha}_j = \vec{e}_j - \vec{e}_{j+1} & \text{for } j=1,2,\cdots n-1 \\ \vec{\alpha}_n = \vec{e}_{n-1} + \vec{e}_n \end{cases}$$



SO(2n)

Hence $\frac{(\alpha_i \cdot \alpha_{j+1})}{\sqrt{(\alpha_i \alpha_i)(\alpha_{j+1}\alpha_{j+1})}} = \frac{-1}{2}$ for $1 \le i \le n-1$, $(\alpha_{n-1}, \alpha_n) = (e_{n-1} - e_n, e_{n-1} + e_n) = 0$

$$\frac{(\alpha_{n-2}, \alpha_n)}{\sqrt{\cdots}} = \frac{(e_{n-2} - e_{n-1}, e_{n-1} + e_n)}{\sqrt{2 \cdot 2}} = -1/2$$

We need to work out the fundamental weight μ^i, satisfying

$$\frac{2(\alpha_i, \mu_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}$$

$$\vec{\mu}_j = \sum_{k=1}^{j} \vec{e}_k \quad \text{for } j = 1, \cdots n-2$$

$$\vec{\mu}_{n-1} = \frac{1}{2}(\vec{e}_1 + \cdots \vec{e}_{n-1} - \hat{e}_n), \quad \vec{\mu}_n = \frac{1}{2}(\vec{e}_1 + \cdots \vec{e}_{n-1} + \hat{e}_n).$$

Test: ① for $j, j' = 1, \cdots n-2$. We have tested in page ③ that

$$\frac{2(\alpha_j, \mu_{j'})}{(\alpha_j, \alpha_j)} = \delta_{jj'}.$$

② Check $\frac{2(\alpha_{n-1}, \mu_{j'})}{(\alpha_{n-1}, \alpha_{n-1})} = ?$ for $j' \le n-2$, we have $\frac{2(\alpha_{n-1}, \mu_{j'})}{(\alpha_{n-1}, \alpha_{n-1})} = 0$

and $$\frac{2(\alpha_{n-1}, \mu_{n-1})}{(\alpha_{n-1}, \alpha_{n-1})} = \frac{2[\hat{e}_{n-1} - \hat{e}_n][\frac{\hat{e}_1}{2} + \cdots \frac{\hat{e}_{n-1}}{2} - \frac{\hat{e}_n}{2}]}{2} = 1$$

$$\frac{2(\alpha_{n-1}, \mu_n)}{(\alpha_{n-1}, \alpha_{n-1})} = \frac{2[\hat{e}_{n-1} - \hat{e}_n][\frac{\hat{e}_1}{2} + \cdots \frac{\hat{e}_{n-1}}{2} + \frac{\hat{e}_n}{2}]}{2} = 0$$

Check $\frac{2(\alpha_n, \mu_{j'})}{(\alpha_n, \alpha_n)} = ?$ for $j' \le n-2$, $\frac{2(\alpha_n, \mu_{j'})}{(\alpha_n, \alpha_n)} = 0$

$$\frac{2(\alpha_n, \mu_{n-1})}{(\alpha_n, \alpha_n)} = \frac{2[e_{n-1} + e_n][\frac{\hat{e}_1 + \cdots \hat{e}_{n-1} - \hat{e}_n}{2}]}{2} = 0$$

$$\frac{2(\alpha_n, \mu_n)}{(\alpha_n, \alpha_n)} = \frac{2[\hat{e}_{n-1} + \hat{e}_n]\frac{1}{2}[e_1 + \cdots \hat{e}_{n+1} + \hat{e}_n]}{2} = 1$$

③ Check $\frac{2(\alpha_j, \mu_{n-1})}{(\alpha_j, \alpha_j)} = ?$ for $j \le n-2 \Rightarrow (\alpha_j, \mu_{n-1}) = 0$

$$\left.\begin{aligned} (\alpha_{n-1}, \mu_{n-1}) &= (\hat{e}_{n-1} - \hat{e}_n, \tfrac{1}{2}(e_1 + \cdots \hat{e}_{n-1} - \hat{e}_n)) = 1 \\ (\alpha_n, \mu_{n-1}) &= (\hat{e}_{n-1} + \hat{e}_n, \tfrac{1}{2}(e_1 + \cdots e_{n-1} - e_n)) = 0 \end{aligned}\right\} \Rightarrow \frac{2(\alpha_j, \mu_{n-1})}{(\alpha_j, \alpha_j)} = \delta_{j,n-1}$$

Check $\frac{2(\alpha_j, \mu_n)}{(\alpha_j, \alpha_j)} = ?$ for $j \le n-2$, $(\alpha_j, \mu_n) = 0$

$$\left.\begin{aligned} (\alpha_{n-1}, \mu_n) &= (e_{n-1} - e_n, \tfrac{1}{2}(e_1 + \cdots e_{n-1} + e_n)) = 0 \\ (\alpha_n, \mu_n) &= (e_{n-1} + e_n, \tfrac{1}{2}(e_1 + \cdots e_{n-1} + e_n)) = 1 \end{aligned}\right\} \Rightarrow \frac{2(\alpha_j, \mu_n)}{(\alpha_j, \alpha_j)} = \delta_{j,n}.$$

Then $\vec{\mu}^{n-1}$ and $\vec{\mu}^{n}$ correspond to spinor representations.

① D^{n-1} : $\mu^{n-1} = \frac{1}{2}(\hat{e}_1 + \hat{e}_2 + \cdots \hat{e}_{n-1} - \hat{e}_n)$

$\swarrow -\alpha_{n-1}$

$\frac{1}{2}(\hat{e}_1 + \cdots - \hat{e}_{n-1} + \hat{e}_n)$

$\swarrow -\alpha_{n-2}$

$\frac{1}{2}(\hat{e}_1 + \cdots - \hat{e}_{n-2} + \hat{e}_{n-1} + \hat{e}_n)$

$\xrightarrow{-\alpha_n} \frac{1}{2}(\hat{e}_1 + \cdots - \hat{e}_{n-2} - \hat{e}_{n-1} - \hat{e}_n)$

$\vdots$

we generates a series doublets and end up with all the weights

$$\boxed{\mu = \frac{1}{2}\sum_{i=1}^{n} \eta_i \hat{e}_i \, , \quad \text{where } \eta_i = \pm 1 ,}$$

but applying α_n's does not change the even and oddness of "$-$" signs, i.e. $\boxed{\prod_{i=1}^{n} \eta_i = -1. \text{ Hence the \# of weights } = 2^n/2 = 2^{n-1} \text{ dim}}$

② D^n : $\mu^n = \frac{1}{2}(\hat{e}_1 + \hat{e}_2 + \cdots \hat{e}_{n-1} + \hat{e}_n)$

$\swarrow -\alpha_n$

$\frac{1}{2}(\hat{e}_1 + \cdots - \hat{e}_{n-1} - \hat{e}_n)$

$\swarrow -\alpha_{n-2}$

$\frac{1}{2}(\hat{e}_1 + \cdots - \hat{e}_{n-2} + \hat{e}_{n-1} - \hat{e}_n)$

$\cdots$

we arrive all the weights

$$\boxed{\mu = \frac{1}{2}\sum_{i=1}^{n} \eta_i \hat{e}_i \quad \text{with } \prod_{i=1}^{n} \eta_i = 1 \Rightarrow \dim = 2^{n-1}.}$$

\# - if n is odd or $SO(4k+2)$, $\Rightarrow$ ~~each of μ_{n-1}~~ ~~representations~~

Reps μ_{n-1} and μ_n are not inversion invariant, and μ_{n-1} and μ_n are related by the inversion. Hence μ_{n-1} and μ_n are complex conjugate to each other. They are complex representations.

if $n=2k$, i.e $SO(2n)=SO(4k)$

Then each of the weight diagrams denoted by μ_{n-1} and μ_n is inversionally invariant. Both μ_{n-1} and μ_n are real / pseudo-real.

⊛ Gamma matrices for SO(2n) group

Check the level n - Gamma matrices

$$\Gamma_1^{(n)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \Gamma_{2\sim 2n}^{(n)} = \begin{pmatrix} 0 & -i\Gamma_i^{(n-1)} \\ i\Gamma_i^{(n-1)} & 0 \end{pmatrix} \quad \Gamma_{2n+1}^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Then $\Gamma_{ab}^{(n)} = -i\,\Gamma_a^{(n)}\Gamma_b^{(n)} = \begin{cases} \begin{pmatrix} \Gamma_{b-1}^{(n-1)} & 0 \\ 0 & -\Gamma_{b-1}^{(n-1)} \end{pmatrix} & \text{for } a=1,\ b=2\sim 2n \\ \begin{pmatrix} \Gamma_{a-1,b-1}^{(n-1)} & 0 \\ 0 & \Gamma_{a-1,b-1}^{(n-1)} \end{pmatrix} & \text{for } 2\leq a<b\leq 2n \end{cases}$

Hence, this 2^n-dimension Rep is reducible to a pair of 2^{n-1} dimensional fundamental spinor Reps. They carry opposite chiral indices ± 1, according to the eigenvalue of Γ_5 defined as $\Gamma_{2k+1}^{k} = \begin{pmatrix} I & \\ & -I \end{pmatrix}$ here.

- How about R-matrix?

For $SO(8k+4)$ and $SO(8k)$, the above construction are based on the level $4k+2$ and $4k$ - Γ-matrices, respectively. The R-matrix is a product of $4k+2$ or $4k$ purely imaginary Γ-matrix, hence

For $n = 4k$, and $4k+2$, the reduced $\Gamma_{ab}^{(n)}$ for $1 \le a < b \le 2n$ are decomposed into diagonal blocks with $\Gamma_a^{(n-1)}$ and $\Gamma_{ab}^{(n-1)}$ matrices. The corresponding R-matrix at $n-1 = 4k \pm 1$ level involves odd number of Γ-matrix (purely imaginary), hence

$$\left.\begin{array}{l} R^{(n-1)} \Gamma_a^{(n-1)} R^{(n-1),-1} = -\Gamma_a^{*(n-1)} \\ \text{and also } R^{(n-1)} \Gamma_{ab}^{(n-1)} R^{(n-1),-1} = -\Gamma_{ab}^{*,(n-1)} \end{array}\right\} \Rightarrow$$

The reduced $n-1$, level $\Gamma_{ab}^{(n-1)}$ for $1 \le a < b \le 2n$ are real or pesuedo-real. $(R^{n-1})^T = R^{(n-1)} (-)^{\frac{n(n-1)}{2}} = \begin{cases} R^{(n-1)} & n = 4k \\ -R^{(n-1)} & n = 4k+2 \end{cases}$

$\Rightarrow$ SO(8k) spiner is real

SO(8k+4) spiner is ~~ps~~ pseudo-real

Another point is that for the level n- R matrices, when n = even, R-matrix itself commutes with $\Gamma_5 = \Gamma_{2n+1}^{(n)}$. Hence R itself is decomposed into diagonal blocks $R^{(n)} = \begin{pmatrix} R^{(n-1)} & \\ & R^{(n-1)} \end{pmatrix}$ for $n = 2k$.

(*) For SO(8k+2) and SO(8k+6)

Since the weight diagrams of μ_{n-1} and μ_n are not inversienally invariant, they are complex conjugate to each other.

For level $n = 4k+1$ and $4k+3$ level Gama matrices

μ_{n-1}: $\Gamma_a^{(n-1)}$, $\Gamma_{ab}^{(n-1)}$ for $1 \le a < b \le 2n-1$

μ_n: $-\Gamma_a^{(n-1)}$, $\Gamma_{ab}^{(n-1)}$

The R-matrix at $n-1 = 4k, 4k+2$ level. satisfying

$$R^{(n-1)} \Gamma_a^{(n-1)} R^{(n-1)-1} = \Gamma_a^{*(n-1)} = -(-\Gamma_a)^{*(n-1)}$$

$$R^{(n-1)} \Gamma_{ab}^{(n-1)} R^{(n-1),-1} = -\Gamma_{ab}^{*(n-1)} = -\left(\Gamma_{ab}^{(n-1)}\right)^*$$

Hence any generaters T in μ_{n-1} $\Rightarrow$

$$R T R^{-1} = -(T'^*), \quad \text{where } T' \text{ in } \mu_n$$

Hence $R e^{-iT} R^{-1} = (e^{-iT'})^*$, i.e μ_{n-1} and μ_n are complex conjugate to each other. In summary:

	spinor		spinor
SO(8k+1)	real	SO(8k+2)	complex
SO(8k+3)	pseudo real	SO(8k+4)	pseudo real
SO(8k+5)	pseudo real	SO(8k+6)	complex
SO(8k+7)	real	SO(8k+8)	real