Lect 13 Representations of Lie algebra

⊛ Fundamental weights

Consider an irreducible representation of a Lie algebra. It's highest weight $|\mu\rangle$ is defined as for any positive root ϕ, such that $\phi + \mu$ is not a weight, i.e.,

$$\boxed{E_{\alpha_i} |\mu\rangle = 0, \ \forall i \text{ where } \alpha_i \text{ is a simple root.}}$$

An irreducible representation can be constructed by applying lowering operators to highest weight states. Use the $q^i - p^i$ languge, $p^i = 0$ for any i.

$$\boxed{\frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} \equiv l_i \geq 0}$$ where l_i are non-negative numbers.

We can make an analogy with crystalline lattice, α_i — unit vectors in the real space, μ's are like receiprocal lattice vector. We define the base vectors of μ as μ_k, such as

$$\boxed{\frac{2(\alpha_i\ \mu_k)}{(\alpha_i, \alpha_i)} = \delta_{ik}, \quad \text{and } \underset{\text{highest weight}}{\mu} = \sum_{i=1}^{m} \underset{\text{non-negative}}{l_i} \mu_i}$$

This kind of weights μ_i are called fundamental weights. A representation can take $\mu = \sum_{i=1}^{m} l_i \mu_i$ as the highest weight state, with $|\mu\rangle = \underbrace{|\mu_1\rangle \otimes \cdot |\mu_1\rangle \cdots}_{l_1} \underbrace{|\mu_m\rangle \cdot |\mu\ldots}_{l_m}$

Theorem: The highest weight states $|\mu\rangle$ for a simple Lie algebra is non-degenerate. The sufficient and necessary conditions for two irreducible representations to be equivalent is that their highest weights are the same.

Proof: ① Assume there exists another highest weight state $|\mu\rangle_2$, and we express $|\mu\rangle_2$ as $\underbrace{E_\lambda \cdots E_\beta E_\alpha}_{\text{linear combinations of}} |\mu\rangle$.

we can shift the positive roots all the way to the right following

$$E_\rho E_\tau = E_\tau E_\rho + [E_\rho, E_\tau], \text{ with } \rho\text{: positive root}, \; \tau\text{: negative root}$$

Then $[E_\rho, E_\tau]$ has a few posibilites: ① zero, ② H_i, → constant acting in $|\mu\rangle$ ③ $E_{\rho+\tau}$. In any case, the second term contains less generaters.

We can repeat the above process, to move E_ρ with ρ positive roots to the right, which annihalate $|\mu\rangle$. If a term has no E_+, then it should not have E_-' type opearators either, otherwise it lowes the weight. Then $|\mu\rangle_2 \propto |\mu\rangle$ up to a constant, hence, no degeneracy.

② If two Reps are equivalent, then their highest weights are equal.

If two Reps have the same highest weight, we can make a correspondence between $|\mu\rangle$ and $|\mu\rangle'$. Then other states obtained by applying generators are also mapping into each other

$$|j\rangle = E_\lambda \cdots E_\beta E_\alpha |\mu\rangle \longleftrightarrow |j\rangle' = E_\lambda \cdots E_\beta E_\alpha |\mu\rangle' .$$

Hence these two Reps are equivalent.

Hence we can use the highest weight to denote the irreducible Reps of Lie algebra.

(*) SU(3) as an example

Simple roots. $\alpha_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ $\qquad A_{ij} = \frac{2(\alpha_i . \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\alpha_2 = (1/2, -\sqrt{3}/2)$

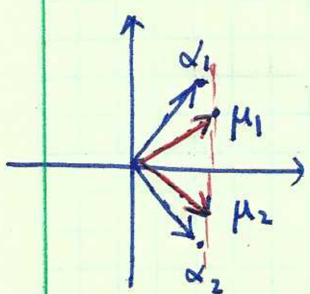
Let me express $\mu_k = \sum_j b_{kj}\, \alpha_j$, then

$$\frac{2(\alpha_i, \mu_k)}{(\alpha_i, \alpha_i)} = \sum_j b_{kj} \frac{2(\alpha_i\, \alpha_j)}{(\alpha_i, \alpha_i)} = \sum_j b_{kj} A_{ji} = \delta_{ki}$$

$$\Rightarrow \quad b_{kj} = (A^{-1})_{kj} \quad \Rightarrow \boxed{\vec{\mu}_k = \sum_j (A^{-1})_{kj}\, \vec{\alpha}_j}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \vec{\mu}_1 = \frac{1}{3}(2\vec{\alpha}_1 + \vec{\alpha}_2) = (1/2, \tfrac{\sqrt{3}}{6})$$

$$\vec{\mu}_2 = \frac{1}{3}(\alpha_1 + 2\vec{\alpha}_2) = (1/2, -\sqrt{3}/6)$$



• Fundamental Reps generated by μ_1 (highest weight)

we can also use the $q_i - P_i$ langage.

$$\frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} = q_i - P_i = 2E_3$$ even if μ is not a highest weight.

We start with the weight μ_1, and we take $\frac{2(\alpha_i, \mu_1)}{(\alpha_i, \alpha_i)}$ for $i=1,2$

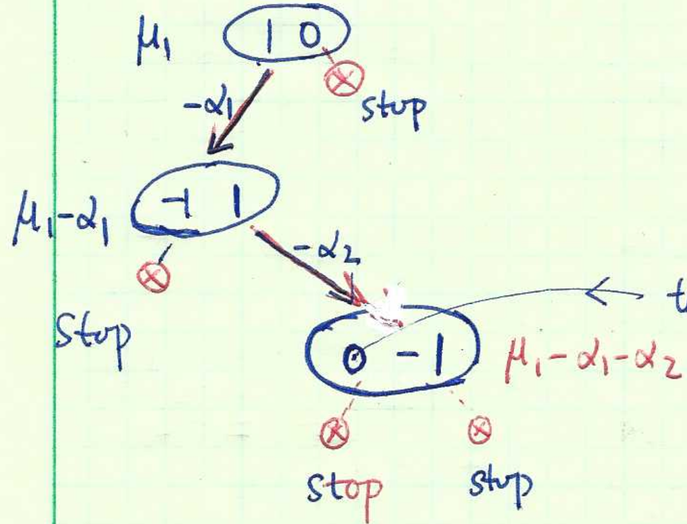
μ_1 (1 0). (highest weight, hence. $P_i = 0$)

Then we start to subtract the simple root: We cannot subtract α_2 since it already ~~starts~~ $2E_3$ (α_2-based) $= 0$, We can only

Subtract α_1. After this, what should we write for $\mu_1 - \alpha_1$?

$$\frac{2(\alpha_i, \mu_1 - \alpha_1)}{(\alpha_i, \alpha_i)} = \frac{2(\alpha_i, \mu_1)}{(\alpha_i, \alpha_i)} - A_{1i},$$ hence, we need to

subtract the first row of A. Then

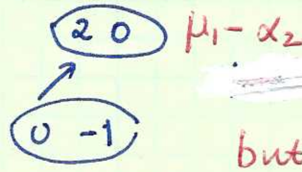


μ_1 and $\mu_1 - \alpha_1$ are α_1 - doublet

$\mu_1 - \alpha_1$ and $\mu_1 - \alpha_1 - \alpha_2$ are α_2 -doublet

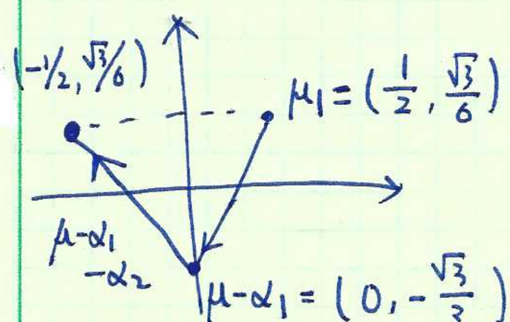
this '0' is an SU(2) highest weight

since if we have,

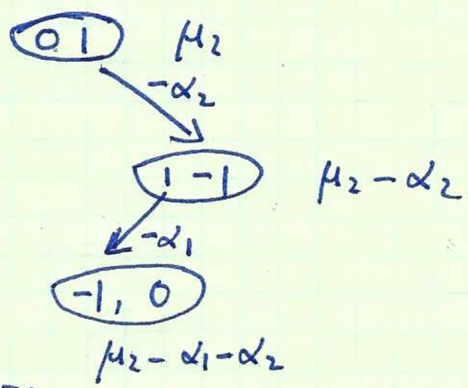


but $\mu_1 - \alpha_2$ is not a weight!

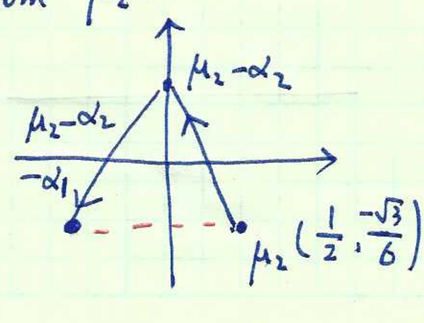
This corresponds to



This is the fundmental spinor Rep.



we can also ~~starts~~ from μ_2:



* We can write down states constructed from the highest weight state $|\mu\rangle$ as $E_{-\alpha^{i_1}} E_{-\alpha^{i_2}} \cdots E_{-\alpha^{i_n}} |\mu\rangle$, where

$E_{-\alpha^i}$'s are lowering operators, and α^i are simple roots.

① two states constructed by two different sets of $E_{-\alpha^i}$'s are orthogonal to each other. It's because their weights are different, and simple roots are linearly independent.

② The norma of the above state

$$\langle\mu| E_{\alpha^{i_n}} \cdots E_{\alpha^{i_1}} E_{-\alpha^{i_1}} \cdots E_{-\alpha^{i_n}} |\mu\rangle$$

we can successively move raising operators to the right. Since $[E_{\alpha^i}, E_{-\alpha^j}] =$ 0 (the difference between two simple roots is not a root), they commute unless the roots are positive-negative paired. Hence, the above equations

$$= \langle\mu| E_{\alpha^{i_1}} E_{-\alpha^{i_1}} \cdots E_{\alpha^{i_n}} E_{-\alpha^{i_n}} |\mu\rangle = \langle\mu| \underbrace{[E_{\alpha^i}, E_{-\alpha^i}]}_{\Pi_i} |\mu\rangle$$

$[E_{\alpha^i}, E_{-\alpha^i}] = \alpha^i \cdot H^i$ $\Rightarrow$ it gives rise to a constant.

③ two states constructed by the same sets of $E_{-\alpha}$'s but they are at different orders. Then these two states are with the same weights. We can calculted their inner product, and perform the Schmidt procedure to build up orthogonal basis.

Then we get degenerate states with the same weights.

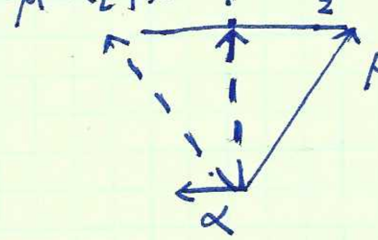
⊛ The Weyl group

If μ is a weight, and $E_3 = \vec{\alpha}\cdot\vec{H}/(\alpha,\alpha)$ is associated with the root $\vec{\alpha}$,

$$\Rightarrow \quad E_3|\mu\rangle = \frac{(\alpha\cdot\mu)}{(\alpha\cdot\alpha)}|\mu\rangle, \quad \text{with } q-p = 2\frac{\alpha\cdot\mu}{(\alpha\cdot\alpha)}$$

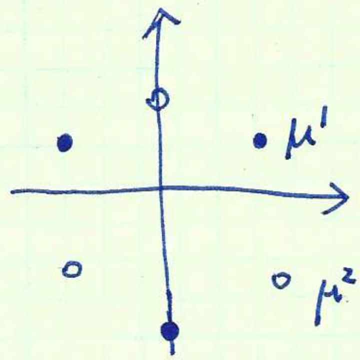
Then $E_3|\mu-(q-p)\alpha\rangle = -\frac{\alpha\cdot\mu}{\alpha\cdot\alpha}|\mu-(q-p)\alpha\rangle$, hence there's a reflection symmetry with respect to $\frac{\mu+\mu-(q-p)\alpha}{2} = \mu - \frac{(q-p)\alpha}{2}$, which is perpendicular to $\vec{\alpha}$. (check $\vec{\alpha}\cdot[\vec{\mu}-\frac{(q-p)\vec{\alpha}}{2}]=0$].

We can combined ~~reflections~~ reflections from other roots, and the set of these operations are the Weyl group.



☆ For example, the Weyl group of $SU(3)$ is S_3

⊛ Complex conjugation



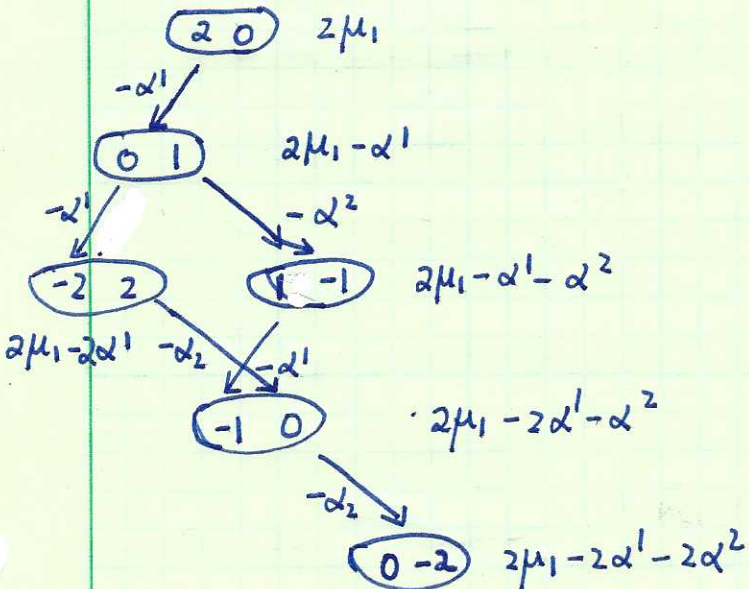
The two representations μ_1 and μ_2 are negatives to each other. They are related by complex conjugation

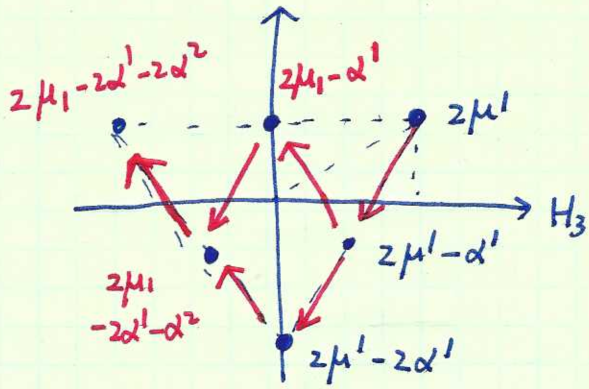
If T_a are ~~gener~~ generators of a representation D, then $-T_a^*$ are generators of D^*. Their eigenvalues $\mu \to -\mu$ and hence weights

Examples of higher representation

- $(n, m) \longleftrightarrow$ highest weight $n\mu_1 + m\mu_2$

Rep $(2,0) \longrightarrow 2\mu^1 = (1, 1/\sqrt{3})$, $\dfrac{2(\alpha_i, 2\mu_1)}{(\alpha_i, \alpha_i)} = \begin{cases} 2 & \text{for } i=1 \\ 0 & i=2 \end{cases}$





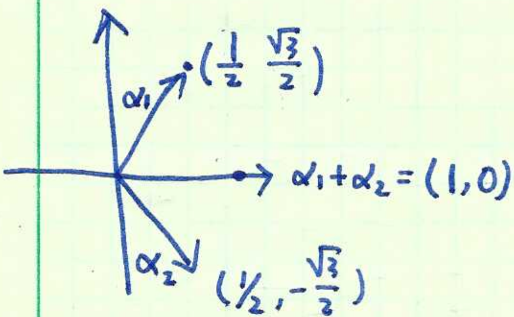
HW: ~~construct~~ the states of the SU(3) Rep (1, 1), whose highest weight $\mu = \mu^1 + \mu^2 = \alpha^1 + \alpha^2$. The zero weight states are doubly degenerate. Please show $|A\rangle = E_{-\alpha^1} E_{-\alpha^2} |\mu^1 + \mu^2\rangle$ and $|B\rangle = E_{-\alpha^1} E_{-\alpha^2} |\mu^1 + \mu^2\rangle$ are linearly indepedent. You may Calculate $\langle A|A\rangle\langle B|B\rangle - \langle A|B\rangle\langle B|A\rangle$. If it's nonzero, it means $|A\rangle$ and $|B\rangle$ are linearly independent.

(*) The dimension of the irreducible Rep μ, (μ the highest weight)

$$d[\mu] = \prod_{\alpha \in \Delta_+} \left[1 + \frac{\vec{\mu}\cdot\vec{\alpha}}{\vec{\rho}\cdot\vec{\alpha}} \right], \quad \text{where } \vec{\rho} = \frac{1}{2}\sum_{\alpha \in \Delta_+} \vec{\alpha}$$

Δ_+ is the set of all positive roots

For example: $\underbrace{\boxed{}\boxed{}\boxed{\cdots}}_{r}$ for $SU(3)$, it's $r\mu_1 = \left(\frac{r}{2}, \frac{\sqrt{3}}{6}r\right) = \mu$



$$\vec{\rho} = \frac{1}{2}\sum_{\alpha \in \Delta_+} \vec{\alpha} = (1, 0)$$

$$\vec{\mu}\cdot\vec{\alpha}_1 = r\left[1/4 + \frac{3}{12}\right] = \frac{r}{2}$$

$$\vec{\mu}\cdot\vec{\alpha}_2 = 0$$

$$\vec{\mu}\cdot(\vec{\alpha}_1+\vec{\alpha}_2) = r/2$$

$$\vec{\rho}\cdot\vec{\alpha}_1 = \frac{1}{2}, \quad \vec{\rho}\cdot\vec{\alpha}_2 = 1/2, \quad \vec{\rho}\cdot(\vec{\alpha}_1+\vec{\alpha}_2) =$$

$$\Rightarrow d = \left[1 + \frac{r/2}{1/2}\right][1]\left[1 + r/2\right] = \frac{[1+r][2+r]}{2}$$

which is the same as $d = \dfrac{\boxed{3}\boxed{4}\boxed{\cdot\cdot}\boxed{r+2}}{\boxed{r}\boxed{r-1}\boxed{\cdot\cdot}\boxed{1}} = \dfrac{(r+2)(r+1)}{2}$

$\circledast$ The 2nd order Casimir $C_2[\mu] = \vec{\mu}\cdot[\vec{\mu}+2\vec{\rho}]$

Proof: Since Casimir is a constant matrix in one representation. Let us evaluate C_2 for the state of $|\vec{\mu}\rangle$

$$C_2 = \sum_{i=1}^{l} H_i^2 + \sum_{\alpha\in\Delta^+}\left[E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha\right]$$

pay attention to the normalization

$[I_i, I_j] = i f_{ij}^{k} I_k$

$\mathrm{Tr}[I_i^{ad} I_j^{ad}] = C_2^{ad}\delta_{ij}$

$\langle E_\alpha | E_\beta\rangle$

$= (C_2^{ad})^{-1}\,\mathrm{Tr}[E_\alpha^{\dagger(ad)} E_\beta^{(ad)}]$

$= \delta_{\alpha\beta}$

$$H_i|\mu\rangle = \mu_i|\mu\rangle$$

$$\Rightarrow \langle\mu|\sum_{i=1}^{l} H_i^2|\mu\rangle = \vec{\mu}\cdot\vec{\mu},$$ ← the classic part

$$E_\alpha|\mu\rangle = 0$$

$$E_\alpha E_{-\alpha} = E_{-\alpha}E_\alpha + [E_\alpha, E_{-\alpha}]$$

$$= E_{-\alpha}E_\alpha + \vec{\alpha}\cdot\vec{H}$$

$$\Rightarrow \langle\mu|E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha|\mu\rangle = \langle\mu|\vec{\alpha}\cdot\vec{H}|\mu\rangle = \vec{\alpha}\cdot\vec{\mu}$$

$$\Rightarrow C_2[\vec{\mu}] = \vec{\mu}\cdot\vec{\mu} + \Big(\sum_{\alpha\in\Delta}\vec{\alpha}\Big)\vec{\mu} = \vec{\mu}\cdot(\vec{\mu}+2\vec{\rho}]$$

For example

$$C_2[\underbrace{\square\square\cdots\square}_{r}] = r\vec{\mu}_1\cdot(r\vec{\mu}_1+\vec{\rho}) = r^2\cdot\frac{1}{3} + r\left(\frac{1}{2}\ \frac{\sqrt{3}}{6}\right)\cdot(1,0)$$

$$= \frac{r^2}{3} + r = r\left[\frac{r}{3}+1\right]$$

(*) Example of SU(N) group

Dynkin diagram $\underset{1}{\circ} - \underset{2}{\circ} - \cdots \underset{N-1}{\circ}$

$$H_1 = T_3 = \frac{1}{2}\begin{bmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{bmatrix}, \quad H_2 = \frac{1}{\sqrt{12}}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & \ddots \end{bmatrix} \cdots \quad H_m = \frac{1}{\sqrt{2m(m+1)}}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -m \end{bmatrix}$$

the weights of the fundementals

$$\begin{cases} \nu^1 = [\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots \left(\frac{1}{2m(m+1)}\right)^{1/2}, \cdots \left(\frac{1}{2(N-1)N}\right)^{1/2}) \\ \nu^2 = [-\frac{1}{2} \; \frac{1}{2\sqrt{3}} \cdots \left(\frac{1}{2m(m+1)}\right)^{1/2} \cdots \left(\frac{1}{2(N-1)N}\right)^{1/2}) \\ \vdots \\ \nu^j = [\underbrace{0 \cdots 0}_{j-2} - \left[\frac{j-1}{2j}\right]^{1/2}, \cdots \left(\frac{1}{2i(i+1)}\right)^{1/2}, \cdots \left(\frac{1}{2(N-1)N}\right)^{1/2}) \\ \vdots \\ \nu^N = [\underbrace{0 \cdots \cdots 0}_{N-2}, -\left(\frac{N-1}{2N}\right)^{1/2}) \end{cases}$$

The simple roots

$$\begin{cases} \alpha^1 = \nu^1 - \nu^2 = [\overbrace{1, 0, \cdots 0}^{N-1 \text{ dimensional}}] \\ \alpha^2 = \nu^2 - \nu^3 = [-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \cdots 0] \\ \alpha^3 = \nu^3 - \nu^4 = [0 - \sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}, 0 \cdots 0] \\ \vdots \\ \alpha^j = \nu^j - \nu^{j+1} = [\underbrace{0, \cdots 0}_{j-2}, -\sqrt{\frac{j-1}{2j}}, \sqrt{\frac{j+1}{2j}}, \cdots 0] \\ \vdots \\ \alpha^{N-1} = \nu^{N-1} - \nu^N = [\underbrace{0, \cdots 0}_{N-3}, -\sqrt{\frac{N-2}{2(N-1)}}, \sqrt{\frac{N}{2(N-1)}}] \end{cases}$$

For simplicity, we use the convention of positive roots by counting non-negative roots from right to left.

Check cartan matrix

$$(\alpha_i \cdot \alpha_i) = 1$$

$$(\alpha_1, \alpha_2) = -1/2, \quad (\alpha_1, \alpha_j) = 0 \quad \text{for } j \geq 3$$

$$(\alpha_2, \alpha_3) = -1/2, \quad (\alpha_2, \alpha_j) = 0 \quad \text{for } j \geq 4$$

$$\vdots$$

$$(\alpha_j, \alpha_{j+1}) = \sqrt{\frac{j+1}{2j}} \, (-) \sqrt{\frac{j}{2(j+1)}} = -1/2, \quad (\alpha_j, \alpha_k) = 0 \quad \text{for } k \geq j+1$$

hence

$$A_{ij} = \frac{2(\alpha_i \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

$$(A^{-1})_{ij} = \frac{1}{N} \begin{array}{c|c} & \begin{array}{ccccccc} 1 & & i & & j & & N-1 \end{array} \\ \hline \begin{array}{c} 1 \\ \\ i \\ \\ j \\ N-1 \end{array} & \left[\begin{array}{ccccccc} 1\cdot(N-1) & & 1\cdot(N-i) & \cdots & 1\cdot(N-j) & \cdots & 1\cdot 1 \\ & \ddots & & & & & \vdots \\ 1\cdot(N-i) & & i(N-i) & \cdots & i(N-j) & \cdots & i\cdot 1 \\ & & & \ddots & & & \vdots \\ 1(N-j) & & i(N-j) & \cdots & j(N-j) & & j\cdot 1 \\ 1\cdot 1, & & i\cdot 1, & \cdots & j\cdot 1 & & (N-1)\cdot 1 \end{array} \right] \end{array}$$

$$\text{or } (A^{-1})_{ij} = \frac{1}{N} \begin{cases} i(N-j) & \text{for } i<j \\ j(N-i) & \text{for } i>j \end{cases}$$

$\boxed{(A^{-1})_{ij} \text{ is a symmetri} \text{ matrix}}$

Then the fundamental weights

$$\vec{\mu}_k = \sum_j (A^{-1})_{kj}\, \vec{\alpha}_j$$

$$\vec{\mu}_1 = \sum_j (A^{-1})_{1j}\, \vec{\alpha}_j = \frac{1}{N}\sum_{j=1}^{N-1}(N-j)\,\vec{\alpha}_j = \frac{1}{N}\sum_{j=1}^{N-1}(N-j)(\vec{\nu}_j - \vec{\nu}_{j+1})$$

$$= \frac{1}{N}\left[(N-1)\vec{\nu}_1 - \vec{\nu}_2 \cdots - \vec{\nu}_N\right] = \vec{\nu}_1 \quad (\text{since } \vec{\nu}_1 + \cdots \vec{\nu}_N = 0)$$

$$\vec{\mu}_2 - \vec{\mu}_1 = \sum_j \left[(A^{-1})_{2j} - (A^{-1})_{1j}\right] \vec{\alpha}_j = \frac{1}{N}\left[-\vec{\alpha}_1 + \sum_{j=2}^{N-1}(N-j)\,\vec{\alpha}_j\right]$$

$$= \frac{1}{N}\left[\vec{\nu}_2 - \vec{\nu}_1 + (N-2)\vec{\nu}_2 - \vec{\nu}_3 \cdots - \vec{\nu}_N\right] = \vec{\nu}_2$$

$\vdots$

$$\vec{\mu}_{i+1} - \vec{\mu}_i = \sum_j \left[(A^{-1})_{i+1,j} - (A^{-1})_{ij}\right] \vec{\alpha}_j = \frac{1}{N}\left[\sum_{j=1}^{i}(-j)\,\vec{\alpha}_j + \sum_{j=i+1}^{N-1}(N-j)\,\vec{\alpha}_j\right]$$

$$= \frac{1}{N}\left[-\sum_{j=1}^{i}(\nu^j - \nu^{j+1})\cdot j + \sum_{j=i+1}^{N-1}(N-j)(\vec{\nu}^j - \vec{\nu}^{j+1})\right]$$

$$= 1/N\left[-\nu^1 - \nu^2 \cdots \nu^i + i\cdot\nu^{i+1} + (N-i-1)\nu^{i+1} - \nu^{i+2} - \cdots \nu^N\right]$$

$$= 1/N\left[-\nu^1 - \nu^2 \cdots - \nu^N + N\nu^{i+1}\right] = \nu^{i+1}$$

$$\Rightarrow \boxed{\begin{cases} \vec{\mu}_1 = \vec{\nu}_1 \\ \vec{\mu}_2 = \vec{\nu}_1 + \vec{\nu}_2 \\ \vdots \\ \vec{\mu}_{N-1} = \vec{\nu}_1 + \vec{\nu}_2 + \cdots \vec{\nu}_{N-1} \end{cases}} \qquad \boxed{\begin{aligned} \vec{\alpha}_1 &= \vec{\nu}_1 - \vec{\nu}_2 \\ \vec{\alpha}_2 &= \vec{\nu}_2 - \vec{\nu}_3 \\ &\vdots \\ \vec{\alpha}_{N-1} &= \vec{\nu}_{N-1} - \vec{\nu}_N \end{aligned}}$$

$$\vec{\nu}_i \cdot \vec{\nu}_j = -\frac{1}{2N} + \frac{\delta}{}$$

check $(\alpha_i, \mu_k) = (\vec{\nu}_i - \vec{\nu}_{i+1}) \sum_{l=1}^{k} \vec{\nu}_l = \frac{1}{2}\sum_{l=1}^{k} \delta_{il} - \delta_{i+1,l} \leftarrow \delta_{i,l-1}$

$$= \frac{1}{2}\left[\cancel{\delta_{i,1}} + \delta_{i,2} - \cancel{\delta_{i,1}} + \delta_{i,3} - \delta_{i,2} + \cdots \delta_{i,k} - \delta_{i,k-1}\right] = \frac{1}{2}\delta_{i,k}$$

hence $2(\alpha_i, \mu_k)/(\alpha_i, \alpha_i) = \delta_{i,k}$

fundamental weight

Rep

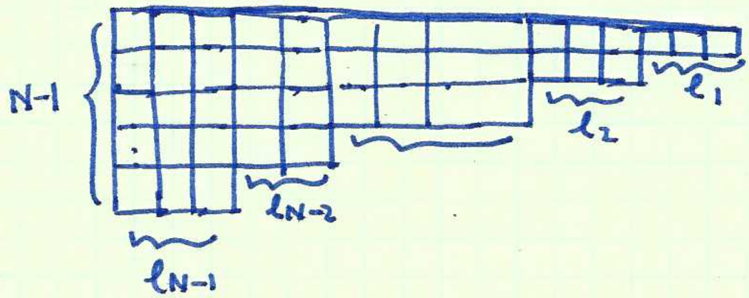
$\vec{\mu}_1$: $\square$

$\vec{\mu}_2$: $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$

$\vdots$

$\vec{\mu}_{N-1}$: $\left.\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \end{array}\right\} N-1$

$\Rightarrow \vec{\mu} = l_1 \vec{\mu}_1 + \cdots \, l_{N-1} \vec{\mu}_{N-1}$ Representation



① Rep with highest weight $\vec{\mu}_1 = \vec{\nu}_1$ — This is the fundamental Rep corresponds to the definition $\vec{\nu}_1 > \vec{\nu}_2 > \cdots > \vec{\nu}_N$, i.e. $\square$.

② Rep with $\vec{\mu}_2 = \vec{\nu}_1 + \vec{\nu}_2$. Consider the fully anti-symmetric Rep $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$

i.e. $a_{i_1 i_2} | \nu_{i_1} \nu_{i_2} \rangle$ with $i_1 \neq i_2$. Hence the highest weight state is $[\,|\nu_1 \nu_2\rangle - |\nu_2 \nu_1\rangle\,]$.

③ : For the rank-r. fully antisymmetric Rep. $\left.\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \end{array}\right\} r.$

the highest weight state is $\varepsilon_{i_1 \cdots i_r} | \nu_{i_1} \cdots \nu_{i_r} \rangle$ with $i_1 \cdots i_r = 1, 2, \cdots r.$

④ $\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline \end{array}$ $\vec{\mu} = r \vec{\mu}_1$

The highest weight state $|\nu_1 \cdots \nu_1\rangle$.