Lect 12. Construction of roots

⊛ Cartan matrix

We have seen that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ plays an important role in the structure of Lie algebra. We define the Cartan matrix for the simple roots α^i as

$$\boxed{A_{ij} = 2\frac{(\alpha^i,\alpha^j)}{(\alpha^j,\alpha^j)}}$$

The off diagonal elements of $A_{ij} < 0$.

$-A_{ij}$ represents the length of the sequence of applying α^j to the simple root α^i: $\alpha^i, \alpha^i+\alpha^j, + \cdots \alpha^i + p\alpha^j$ where $p = -A_{ij}$, $q = 0$.

The diagonal matrix element $A_{ii} = 2$, which can be interpreted as

$-\alpha^i = \alpha^i - 2\alpha^i, \ 0, \ \alpha^i$ with $p = 0$, $q = A_{ii} = 2$.

Now consider any positive root $\phi = \sum_i k_i \alpha^i$ (i is the index of simple roots, k_j non-negative number). Apply α^j to ϕ to arrive at a series of $SU(2)$ multiplets under $E_\pm = \frac{E_{\pm\alpha^j}}{(\alpha^j,\alpha^j)}$ and $E_3 = \vec{\alpha}^j\cdot\vec{H}/(\alpha_j,\alpha_j)$

$$\phi - q^j\alpha^j, \ \ldots \ \phi-\alpha^j, \ \phi, \ \phi+\alpha^j, \ \ldots \ \phi + p^j\alpha^j$$

$$2E_3|\phi\rangle = 2\frac{\alpha^j\cdot H}{(\alpha^j,\alpha^j)}|\phi\rangle = 2\frac{(\alpha^j\cdot\phi)}{(\alpha^j\cdot\alpha^j)}|\phi\rangle = (q^j - p^j)|\phi\rangle$$

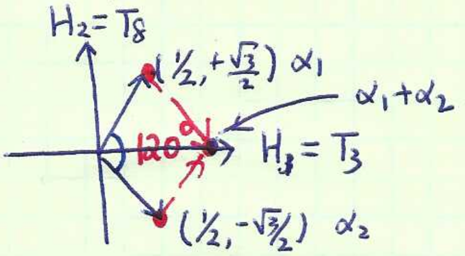
$$\Rightarrow \quad q^j - p^j = \frac{2(\alpha^j,\phi)}{(\alpha^j,\alpha^j)} = \sum_{i=1}^{l} k_i A_{ij},$$

where l is the rank of Cartan subalgebra.

The cartan matrix element A_{ij} tells how the simple roots fit into the SU(2) representations generated by other simple roots.

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad \xleftrightarrow{-A_{ij} = p} \quad \alpha^i, \quad \alpha^i + \alpha^j, \quad \ldots, \quad \alpha^i + p\alpha^j$$

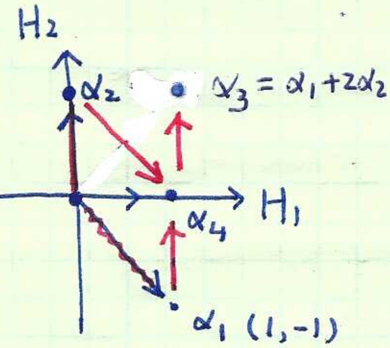
For example: SU(3)



$\alpha_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\alpha_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

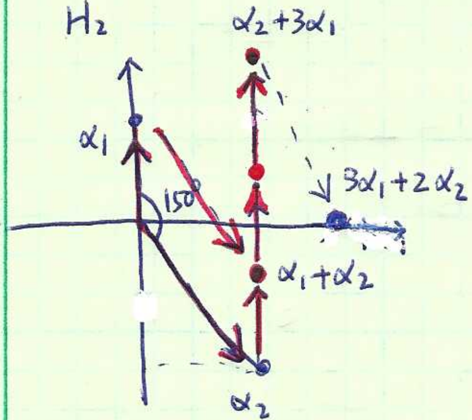
SU(3)



$\alpha_1 = (1,-1)$, $\alpha_2 = (0,1)$

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

SO(5) or Sp(4)



$\alpha_1 = (0,1)$, $\alpha_2 = (\frac{\sqrt{3}}{2}, -\frac{3}{2})$

α_1: short, α_2: long

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

-1 → start from α_1 apply α_2

-3 ↙ A_{21} starting fr α_2, apply α_1

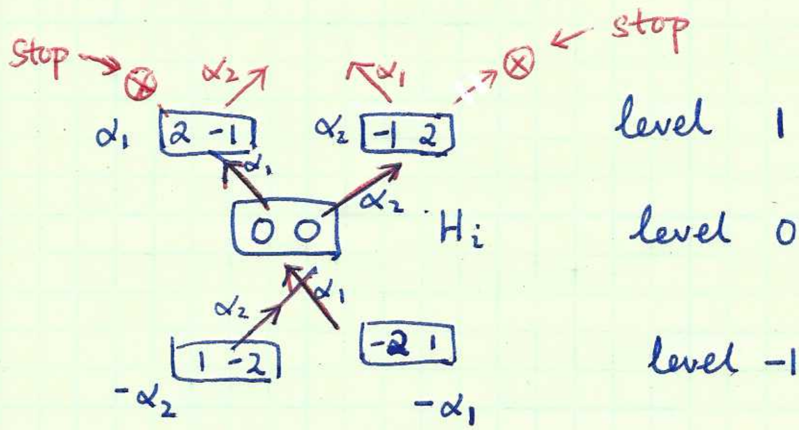
Please note: Some text books define $A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

(*) Cartan matrix provides a convenient way to calculate $2E_3$ of each root in the chain of the $SU(2)$ multiplet generated by another simple root. Let's take $SU(3)$ roots as an example

① level one ($k=1$) roots,

put the first row of A_{ij} into a box : α_1 $\boxed{2\ -1}$,

"2" is the value of $2E_3$ in the root chain generated by appling α_1. hence α_1 is the high end of this chain. We can add the level zero ($k=0$), and level -1 (negative roots) to show the entire chain. "-1" is the value of $2E_3$ in the root chain generated by appling α_2, and α_1 is at its low end. Similarly analysis can be done for the box α_2 $\boxed{-1\ 2}$.



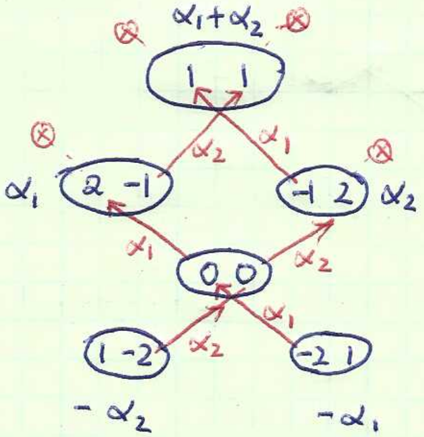
② level two ($k=2$) roots

From $\alpha_1 \xrightarrow{\alpha_2} \alpha_1+\alpha_2$ and from $\alpha_2 \xrightarrow{\alpha_1} \alpha_1+\alpha_2$, they end up with $\alpha_1+\alpha$ what should be written in the box?

Suppose a root ϕ has q^j-p^j by appling α_j. now we consider $\phi+\alpha_i$, then $q'^j-p'^j = q^j-p^j+\frac{2(\alpha_i\cdot\alpha_j)}{(\alpha_j\cdot\alpha_j)} = q^j-p^j+A_{ij}$

i.e. we take the box of ϕ, and add the i-th row of the cartan matri

⊗ means stop.



level 2

level 1 — ($q=0$) — simple roots: lines represen[t] adding different simple roots.

H_i, level 0

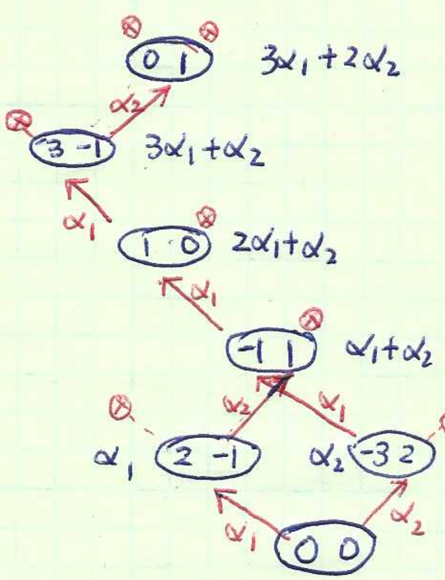
level -1 — From simple roots, we can only go upward.

The line $\alpha_1 : -1 \xrightarrow{\alpha_2} \alpha_1+\alpha_2 : 1$

just view as an S_z-sequence from $-1/2$ to $1/2$.

it stops at $1/2$.

(*) a more complicated example G_2.



level 5

level 4

level 3 → from $2\alpha_1+\alpha_2$, we cannot add / subtract α_2, otherwise we end up with $2\alpha_1$. But we have all level 2 roots, and $2\alpha_1$ is not in.

level 2

level 1

— level 0 H_i

in total, there are 12 roots, and 6 are positive roots.

HW: work out all the root structure of the $Sp(4)$, or isomorphically $SO(5)$ root diagrams. We set $\alpha_1 = (1\ -1)$, $\alpha_2 = (0, 1)$, and $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

⊛ Constructing the G_2-algebra

We have two generators $E_{1+} = E_{\alpha_1}$ $\quad E_0 = \vec{\alpha}_1 \cdot \vec{H}$, $\quad E_{1-} = E_{-\alpha_1}$

$$E_{2+} = \frac{E_{\alpha_2}}{\sqrt{3}}, \quad E_{20} = \vec{\alpha}_2 \cdot \vec{H}, \quad E_{2-} = \frac{E_{-\alpha_2}}{\sqrt{3}}$$

Let's start with $|E_{\alpha_2}\rangle$, which is $|3/2, -3/2\rangle$ (level 1)

$$|[E_{\alpha^1}, E_{\alpha^2}]\rangle = E_{1+}|E_{\alpha_2}\rangle = \frac{1}{\sqrt{2}}\sqrt{(3/2+3/2)(3/2-3/2+1)}\ |3/2\ -1/2\rangle = \sqrt{3/2}\ |E_{\alpha_1+\alpha_2}\rangle$$

hence $\underline{E_{\alpha_1+\alpha_2} = \sqrt{\tfrac{2}{3}}\,[E_{\alpha_1}, E_{\alpha_2}]}$

$$|[E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]\rangle = \sqrt{3/2}\ E_{1+}|3/2, -1/2\rangle = \sqrt{3/2}\sqrt{1/2}\ \sqrt{(3/2+1/2)(3/2-1/2+1)}\ |3/2, 1/2\rangle$$

$$= \sqrt{3}\ |3/2, 1/2\rangle = \sqrt{3}\ |E_{2\alpha_1+\alpha_2}\rangle$$

or $\underline{E_{2\alpha_1+\alpha_2} = \frac{1}{\sqrt{3}}[E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]}$

$$|[E_{\alpha_1}, [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]]\rangle = \sqrt{3}\cdot\sqrt{1/2}\ \sqrt{(3/2-1/2)(3/2+1/2+1)}\ |3/2\ 3/2\rangle$$

$$= \sqrt{9/2}\ |E_{3\alpha_1+\alpha_2}\rangle$$

$\underline{E_{3\alpha_1+\alpha_2} = \sqrt{2/9}\ [E_{\alpha_1} [E_{\alpha_1} [E_{\alpha_1} E_{\alpha_2}]]]}$

treat : $|E_{3\alpha_1+\alpha_2}\rangle = |1/2\ -1/2\rangle$

$$|[E_{\alpha_2}, [E_{\alpha_1} [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]] = \sqrt{3}\ E_{2+}|1/2\ -1/2\rangle = \frac{\sqrt{3}}{\sqrt{2}}\sqrt{9/2}\,|1/2\ 1/2\rangle$$

$\sqrt{9/2}$

$$= \frac{3}{2}\sqrt{3}\ |1/2\ 1/2\rangle = \frac{3\sqrt{3}}{2}\ |E_{3\alpha_1+2\alpha_2}\rangle$$

$$\Rightarrow \quad E_{3\alpha_1+2\alpha_2} = \frac{2}{3\sqrt{3}}\left[E_{\alpha_2}, \left[E_{\alpha_1}, \left[E_{\alpha_1}, \left[E_{\alpha_1}\ E_{\alpha_2}\right]\right]\right]\right]$$