Lect 11 Classification of Simple Lie algebra

Recap ① Lie algebra $[I_i, I_j] = i f_{ij}^k I_k$ and f_{ij}^k are fully anti-sym in terms of ijk. Then in the adjoint Rep., we have

$$\mathrm{Tr}[I_i^{(ad)} I_j^{(ad)}] = C_2 \delta_{ij}$$ where C_2 is the Casimir.

② For the Cartan subalgebra, we choose the basis H_i $(i=1, \cdots l)$ as the maximally intercommutable subset of I_i. The roots are define

$$[H_i, E_\alpha] = \alpha_i E_\alpha,$$ where $\vec{\alpha} = (\alpha_1, \cdots \alpha_l)$ is the root vector

and E_α is normalized according to the inner product

$$\langle E_\alpha | E_\beta \rangle = C_2^{-1} \mathrm{Tr}[E_\alpha^{\dagger(ad)} E_\beta^{(ad)}] = \delta_{\alpha\beta}$$

③ we have

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ \Sigma \alpha_i H_i & \text{if } \alpha = -\beta \\ 0 & \text{if } \alpha+\beta \text{ is not a root} \end{cases}$$

In the following, we will show the strong constraint due to above condition to the structure of root vectors in the root diagram.

§: SU(2): from each pair of roots $E_{\pm\alpha}$

Note: if $\alpha \to -\alpha$
$E_3 \to -E_3$
$E_\pm \to E_\mp$
SU(2) relation maintains

① Consider $E_\pm \equiv |\alpha|^{-1} E_{\pm\alpha}$, $E_3 \equiv |\alpha|^{-2}\, \alpha \cdot H$,

where $|\alpha| = \sqrt{\sum \alpha_i^2}$, $\alpha \cdot H = \sum_{i=1}^{r} \alpha_i H_i$

Exercise: Prove $E_\pm$, E_3 for an SU(2) algebra satisfying

$$[E_3, E_\pm] = \pm E_\pm, \qquad [E_+, E_-] = E_3. \to \begin{cases} E_\pm : \frac{1}{\sqrt{2}}(S_x \pm i S_y) \\ E_3 : S_z \end{cases}$$

② A root vector $\vec{\alpha}$ corresponds to a unique generator

Suppose there are two generaters E_α and E'_α, and their superposition should also correspond to the $\vec{\alpha}$. We can do linear combinations such that $\langle E_\alpha | E'_\alpha \rangle = C_2^{-1} \operatorname{Tr}[E_\alpha^{+(ad)} E_\alpha^{(ad)\prime}] = C_2^{-1} \operatorname{Tr}[E_{-\alpha} E'_\alpha] = 0$

$$E_3 |E'_\alpha\rangle = |\alpha|^{-2} \alpha_i H_i |E'_\alpha\rangle = |\alpha|^{-2} \sum_i \alpha_i^2 |E'_\alpha\rangle = |E'_\alpha\rangle$$

hence the E_3-eigenvale of $|E'_\alpha\rangle$ is 1.

One the other hand: we can also prove $E_- |E'_\alpha\rangle = 0$.

E_- is proportion to a root $E_{-\alpha}$, hence

$E_- |E'_\alpha\rangle$ must belong to the Cartan subalgebra

$$\langle H_i | E_- | E'_\alpha\rangle = C_2^{-1} \operatorname{Tr}[H_i^{ad} [E_-^{ad}, E_\alpha^{\prime ad}]]$$

$$= -C_2^{-1} \operatorname{Tr}[E_-^{ad} [H_i^{ad}, E_\alpha^{\prime ad}]] = -\alpha_i C_2^{-1} \operatorname{Tr}[E_\alpha^{\prime ad} E_-^{ad}] = 0$$

for all i. $\Rightarrow \boxed{E_- |E'_\alpha\rangle = 0,}$

But a state annillated by E_- cannot be E_3 eigenstate with a positive eigenvalue, hence $|E'_\alpha\rangle$ does not exist.

(*) If $\vec{\alpha}$ is a root vector, then along this direction, except $-\vec{\alpha}$, there are no other roots.

Proof: Suppose $k\vec{\alpha}$ ($k \neq \pm 1$) is also a root, which is associated with a generator $E_{k\vec{\alpha}}$. Then $[E_3, E_{k\vec{\alpha}}] = |\alpha|^{-2} [\alpha \cdot H, E_{k\vec{\alpha}}] = \frac{k}{|\alpha|^2} |\alpha|^2 E_{k\vec{\alpha}}$

$= k E_{k\vec{\alpha}}$

then $2k$ must be an integer since k is E_3 eigenvalue.

① If k is an integer, $E_{k\vec{\alpha}}$ is a part of a representation containing another generator with root $\vec{\alpha}$. (we can apply $E_\pm$ to form such a sequence). $E_{\vec{\alpha}}$ itself must not be in such a sequence, since $E_\pm \sim E_{\pm\vec{\alpha}}$ up to a constant. Since we have proved, $\vec{\alpha}$ can only correspond to one root, this will be in a contradiction.

② If k is a half integer, then the sequence of $\cdots E_{(k-1)\vec{\alpha}}$ $E_{k\vec{\alpha}}$, $E_{(k+1)\vec{\alpha}} \cdots$ must contain a state with the root $\vec{\alpha}/2$. Then define $\vec{\beta} = \vec{\alpha}/2$, and $\vec{\alpha} = 2\vec{\beta}$, hence $E_{\vec{\alpha}}$ will correspond to $E_{k\vec{\beta}}$ with $k=2$, which is impossible as just analyzed!

§: "Master formula"

Consider a state belonging to a representation D with a weight μ

$$E_3 \,|\mu, D\rangle = \frac{\alpha \cdot H}{|\alpha|^2} |\mu, D\rangle = \frac{\alpha \cdot \mu}{|\alpha|^2} |\mu, D\rangle$$

Hence $\boxed{2\frac{\alpha \cdot \mu}{|\alpha|^2} = \text{integer}}$

$|\mu, D\rangle$ can always be written as a superposition of states transforming according to the $SU(2)$ $E_\pm$, E_3. Assume the highest spin involved is j. Then there are integer p such at

$$(E^+)^p |\mu D\rangle \neq 0 \quad \text{but} \quad (E^+)^{p+1} |\mu, D\rangle = 0$$

Then the eigenvalue of E_3 for the state of $(E^+)^p |\mu D\rangle$, should be j,

$$\Rightarrow \boxed{\frac{\alpha \cdot (\mu + p\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + p = j}$$

Likewise, there are also non-negative integer q such that

$$(E^-)^q |\mu D\rangle \neq 0 \quad \text{but} \quad (E^-)^{q+1} |\mu D\rangle = 0$$

hence $$\frac{\alpha \cdot (\mu - q\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} - q = -j$$

$$\Rightarrow \frac{2\alpha \cdot \mu}{\alpha^2} + p - q = 0 \quad \text{or} \quad \boxed{\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{p-q}{2}}$$

ceterainly, we can apply it to the root ~~structure~~ of Lie algebra, we have the following theorem:

> If α and β are two roots of a semi-simple Lie algebra, then
>
> $$\Gamma(\alpha/\beta) \equiv \frac{2\alpha \cdot \beta}{\beta \cdot \beta} = \text{integer, and}$$
>
> $\alpha - \Gamma(\alpha/\beta)\beta$ is also a root.

We just need to start with E_α, and apply E_β and $E_{-\beta}$ to form a chain of roots

$$E_{\alpha - q\beta}, \cdots \quad E_{\alpha-\beta} \quad E_\alpha \quad E_{\alpha+\beta} \cdots \quad E_{\alpha+p\beta}$$

They are $SU(2)$ multiplets generated by $E_{\pm} = \frac{1}{|\beta|} E_{\pm\beta}$, and $E_3 = \frac{1}{|\beta|^2} \beta_j \cdot H_j$. Then $\frac{2\alpha\cdot\beta}{\beta^2} = q - p$, and $-p \leq q - p \leq q$

or $-q \leq -\Gamma(\alpha/\beta) \leq p$, hence $\alpha - \Gamma(\alpha/\beta)\beta$ is inside the chain and $\vec{\alpha} - \Gamma(\alpha/\beta)\vec{\beta}$ is a root.

(*) Angles between roots

if we switch the positions between α and β, we also have

$$\beta\cdot\alpha/\alpha^2 = -\tfrac{1}{2}(p'-q') \leftarrow \alpha\cdot\beta/\beta^2 = -\tfrac{1}{2}(p-q)$$

$$\Rightarrow \cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4}$$

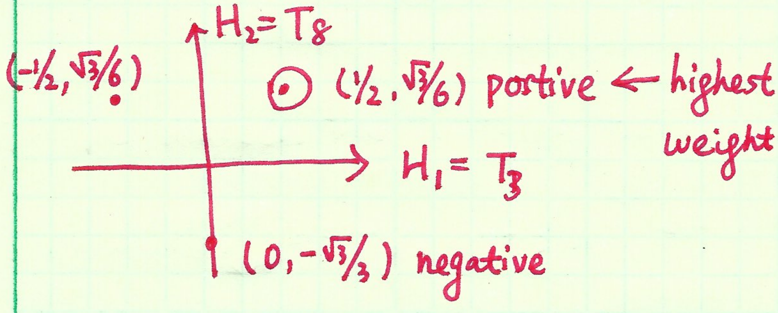
$\cos^2\theta_{\alpha\beta} = 1 \Rightarrow \theta_{\alpha\beta} = 0$, (impossible since only one root is alowed along this direction.

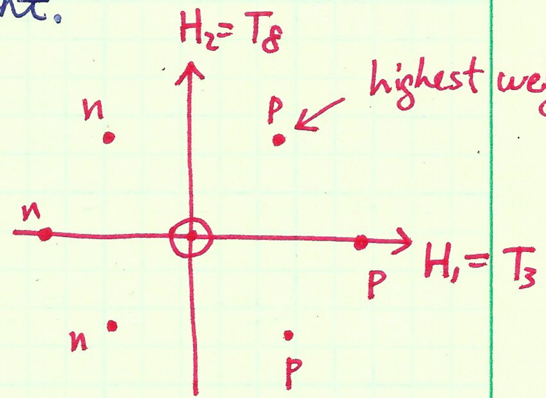
$\theta_{\alpha\beta} = 0$, it's just $\beta = -\alpha$.

$\cos^2\theta =$	0	$\Rightarrow$	$\theta = 90^\circ$
	$1/4$		60° or 120°
	$1/2$		45° or 135°
	$3/4$		30° or 150°

※ Positive weights

In an arbitany basis of Cartan subalgebra, $\vec{\mu} = (\mu_1 \cdots \mu_\ell)$ – the weight vector. We say $\vec{\mu}$ is positive if the first nonzero component > 0, otherwise it's negative. If $\vec{\mu} - \vec{\nu}$ is positive, then we say $\vec{\mu} > \vec{\nu}$, Such that we can define the highest weight.





※ Simple roots

we can define positive roots in the same way as we define positive weight. (roots can be viewed as the weights of the adjoint representation).

Simple roots : Positive roots that cannot be written as sum of other positive roots with positive coefficients.

① If α and β are simple roots, then $\alpha - \beta$ is not a root.

② Ex: please prove it. Hint $\alpha = \beta + (\alpha - \beta)$, or $\beta = \alpha + (\beta - \alpha)$.

② Since $\alpha - \beta$ is not a root, hence the sequence

$$\alpha \quad \alpha+\beta, \cdots \quad \alpha + p\beta \quad \Rightarrow \quad E_{-\beta}|E_\alpha\rangle = [E_\beta \; E_\alpha] = 0$$

$$\beta \quad \beta+\alpha \cdots \quad \beta + p'\alpha \qquad E_{-\alpha}|E_\beta\rangle = [E_{-\alpha} \; E_\beta] = 0$$

Hence $\frac{\alpha\cdot\beta}{\alpha^2} = -\frac{1}{2}(p'-q')$ $\frac{\beta\alpha}{\beta^2} = -1/2\,(p-q)$

with $q'=0$ and $q=0$

$\Rightarrow \cos\theta_{\alpha\beta} = -\frac{\sqrt{pp'}}{2} \le 0$, $\beta^2/\alpha^2 = p'/p$

$\Rightarrow \boxed{\pi/2 \le \theta < \pi}$ → $\theta \neq \pi$. since α and β are positive.

③ Simple roots are linearly independent

HW: prove it , Georgi P106.

④ All positive roots can be written a linear combination

$\phi = \sum_\alpha k_\alpha \alpha$, such that k_α is nonnegative.

Proof: if ϕ is simple, then it's done. If not, by definition, it can be decomposted into a sum of two positive roots. This splitting can be continued until simple roots are reached!

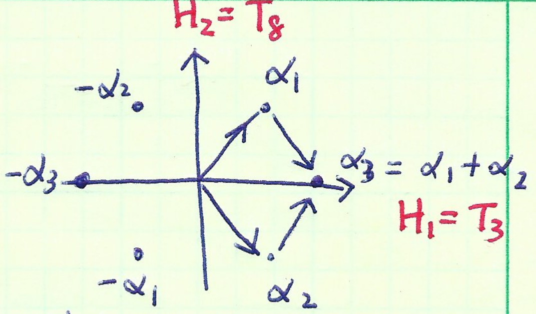
⑤ Simple roots are also complete, hence the # of simple roots = the rank of Cartan subalgebra

HW: prove it Georgi P106

Example SU(3) roots

① Two simple roots

$\alpha_1 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$, $\alpha_2 = (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})$



then

$E_{\alpha_2}|\alpha_1\rangle$, i.e. $\alpha_1 \quad \alpha_1+\alpha_2 ; \cdots \quad \alpha_1 + p\alpha_2 \Rightarrow q = 0$

$$\frac{\vec{\alpha_2}\cdot\vec{\alpha_1}}{|\alpha_2|^2} = -\frac{(p-q)}{2} \Rightarrow \frac{(-\tfrac{1}{2})}{1} = -\frac{p}{2} \Rightarrow p = 1$$

hence there are only $\alpha_1, \alpha_1+\alpha_2$. hence $\alpha_1 + 2\alpha_2$ is not a root.
Similarly $2\alpha_1 + \alpha_2$ is not a root either.

Actually, we can construct the entire algebra:

Consider the chain $\alpha_2, \alpha_2 + \alpha_1$

$$E_+ = \frac{1}{|\alpha_1|} E_{\alpha_1} = E_{\alpha_1} \qquad E_- = E_{-\alpha_1}, \qquad E_3 = \frac{1}{|\alpha_1|^2}\alpha_1\cdot H = \alpha_1 \cdot H$$

$$E_+|E_{\alpha_2}\rangle \rightarrow E_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\;|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$= \frac{\eta}{\sqrt{2}}|E_{\alpha_3}\rangle$, where η is a phase, by convetion set to be 1.

i.e $\Rightarrow |E_{\alpha_3}\rangle = \sqrt{2}\,E_+|E_{\alpha_2}\rangle$ i.e $\boxed{E_{\alpha_3} = \sqrt{2}\,[E_{\alpha_1}, E_{\alpha_2}]}$

Hence we have expressed the positive root E_{α_3} as commutater of E_{α_1}, E_{α_2}
other commutaters can be calculated by using Jacobi identity

$$[E_{-\alpha_1}, E_{\alpha_1+\alpha_2}] = \sqrt{2}\,[E_{-\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]] = \sqrt{2}\,[[E_{-\alpha_1}, E_{\alpha_1}], E_{\alpha_2}]$$

$$= \sqrt{2}\,[-\alpha_1\cdot H, E_{\alpha_2}] = -\sqrt{2}\,\alpha_1\cdot\alpha_2\, E_{\alpha_2} = \frac{1}{\sqrt{2}} E_{\alpha_2}$$

This is consistent with $\alpha_2 \xleftarrow{-\alpha_1} \alpha_2 + \alpha_1 = \alpha_3$

$$E_-|E_{\alpha_3}\rangle \rightarrow E_-|\tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}-\tfrac{1}{2}\rangle \Rightarrow E_-|E_{\alpha_3}\rangle = \frac{1}{\sqrt{2}}|E_{\alpha_2}\rangle$$

$$\Rightarrow [E_-, E_{\alpha_3}] = \frac{1}{\sqrt{2}} E_{\alpha_2} \quad \text{or} \quad \boxed{[E_{-\alpha_1}, E_{\alpha_1+\alpha_2}] = \frac{1}{\sqrt{2}} E_{\alpha_2}}$$

then $[E_{-\alpha_2}, E_{\alpha_1+\alpha_2}] = \sqrt{2}\,[E_{-\alpha_2}\ [E_{\alpha_1}, E_{\alpha_2}]] = \sqrt{2}\,[E_{\alpha_1}, [E_{-\alpha_2}, E_{\alpha_2}]]$

$$= \sqrt{2}\,[E_{\alpha_1}, -\alpha_2 \cdot H] = \sqrt{2}\,[\alpha_2 \cdot H, E_{\alpha_1}] = \sqrt{2}\,\alpha_2 \cdot \alpha_1 E_{\alpha_1} = -\frac{1}{\sqrt{2}} E_{\alpha_1}$$

Please notice the "−" sign.

This is because when we define E_{α_3}, we follow the chain $\alpha_2, \alpha_2+\alpha_1$ if we use the sign convention $\eta = 1$ for $\alpha_1, \alpha_1+\alpha_2$, we would end up with $E_{\alpha_3} = \sqrt{2}\,[E_{\alpha_2}, E_{\alpha_1}]$, which is up to a sign.

⊛ Dykin diagram

We will see later that the lengths of the simple roots can at most have two posibilities. We use ○ for long simple roots and ● for short ones. If all simple roots are with the same lengths, then we use ○.

Recal the relation between two simple roots α and β.

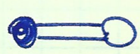
$$\begin{matrix} \alpha,\ \alpha+\beta,\ \cdots\ \alpha+p\beta \\ \beta\ \ \beta+\alpha,\ \cdots\ \beta+p'\alpha \end{matrix} \Rightarrow \begin{matrix} \beta\cdot\alpha/\beta^2 = -\frac{p}{2} \\ \alpha\cdot\beta/\alpha^2 = -\frac{p'}{2} \end{matrix} \Rightarrow \frac{\alpha^2}{\beta^2} = p/p'$$

and $\cos\theta_{\alpha\beta} = \sqrt{pp'}/2$

$\theta_{\alpha\beta}$	$\cos^2\theta$	p ○	p' ●	$\lvert\alpha\rvert/\lvert\beta\rvert$
150°	3/4	3	1	$\sqrt{3}$
135°	1/2	2	1	$\sqrt{2}$
120°	1/4	1	1	1

ratio $\lvert\alpha\rvert/\lvert\beta\rvert$ v.s $\theta_{\alpha\beta}$

 for $\theta = 150°$

 for $\theta = 135°$

 for $\theta = 120°$

no line for $\theta = 90°$

What kind of Dykin diagrams are allowed?

① Dynkin diagram are connected

Proof: If a Dynkin diagram is divided into A and B parts.

~~We know~~: Suppose simple roots $\alpha \in A$, $\beta \in B$, then we know $\alpha - \beta$ is not a root. Since α and β form an angle of $90°$, $\alpha + \beta$ is not a root either, or $p = p' = 0 \Rightarrow \alpha, \beta$ commute.

$\Rightarrow$ The Lie algebra is divided into 2 commutable parts, which is no longer simple Lie algebra.

② Dykin diagram does not ~~ct~~ contain rings.

Proof: if the simple roots $u_1, \cdots u_n$ form a ring. No diagonal lines

$$\vec{a} = \sum_{j=1}^{n} u_j \neq 0$$

but $$|\vec{a}|^2 = \sum_{j=1}^{n} |u_j|^2 + 2 \sum_{j=1}^{n} \vec{u}_j \cdot \vec{u}_{j+1}$$

($\vec{u}_i \cdot \vec{u}_j = 0$ if $i\,j$ are not along an edge)

$$= \sum_{j=1}^{n} |u_j|^2 \left[1 + \frac{2\vec{u}_j \cdot \vec{u}_{j+1}}{|u_j|^2} \right]$$

since $\vec{u}_j \cdot \vec{u}_{j+1} < 0$ and $2\vec{u}_j \cdot \vec{u}_{j+1} / |u_j|^2$ is an integer

hence $1 + 2\vec{u}_j \cdot \vec{u}_{j+1} / |u_j|^2 \leq 0 \Rightarrow |\vec{a}|^2 \leq 0$, which is in contradiction!

③ For each simple roots, there are no more than three lines from it.

Proof: Consider a simple root r, connecting to n simple roots $u_1, \cdots u_n$. And $u_1, \cdots u_n$ do not connect. According to the

$$\begin{cases} r \quad r+u \cdots \quad r+p_u u \\ u \quad u+r, \cdots \quad u+p'_u r \end{cases} \qquad \cos\theta_{ur} = \sqrt{pp'}/2,$$

there are always $p=1$ or $p'=1$

According to the convention of # of lines, we have the total # of lines $= \sum_{j=1}^{n} p_{u_j} p'_{u_j}$, and $\vec{u}_j \cdot \vec{u}_{j'} = 0$

Since $\vec{u}_j$'s are orthogonal to each other, and u_j's are incomplete,

Hence $|r|^2 \geq$ the sum of square of $\vec{r}$'s projection along u_j, i.e

$$|r|^2 > \sum_{j=1}^{n} (r\cdot u_j)^2/|u_j|^2 = \frac{|r|^2}{4}\sum_{j=1}^{n} \frac{2r\cdot u_j}{|r|^2}\cdot\frac{2u_j\cdot r}{|u_j|^2}$$

$$\left.\begin{aligned} &\frac{2r\cdot u_j}{|r|^2} = q-p \\ &q=0, \text{ since } r, u_j \text{ are simple roots} \\ &r\cdot u_j \text{ not orthogonal} \Rightarrow p\geq 1 \end{aligned}\right\} \Rightarrow \frac{2r\cdot u_j}{|r|^2} \leq -1$$

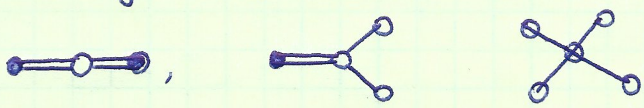
Similarly $2u_j\cdot r/|u_j|^2 \leq -1$

$$\Rightarrow |r|^2 > \frac{|r|^2}{4}\cdot n \quad \Rightarrow \quad n<4, \text{ i.e. } n=0,1,2,3.$$

Hence: Only one Dynkin digram contains the triple line, i.e the G_2 , Ex: prove it!

③ Dynkin digrams containing double line/lines.

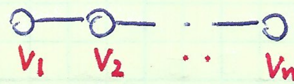
Apparently, the following diagrames are illegeal



They violet the maximal number of lines ≤ 3.

if we exchange ● and ○, they're not allowed either.

But if we insert roots connected by single lines,



will it help?

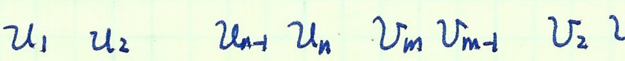
- Since $v_1, \cdots v_m$ are connected by single lines, their lengths are equal.

$$\vec{v} = \sum_{j=1}^{m} \vec{v}_j \Rightarrow |v|^2 = \sum_{j=1}^{m} |v_j|^2 + 2\sum_{j=1}^{m-1} v_j \cdot v_{j+1} \leftarrow 120^\circ$$

$$= m v^2 - (m-1) v^2 = v^2$$

We can use v to replace r. in the part ②, then the proof in ② applies. The total # of lines connetting o—o⋯o should not exceed three.

Hence, the Dynkin diagram with a double line can only be in the configuration



v_k's lengeth v

u_i's length $\sqrt{2}\, v$

Define $\vec{u} = \sum_{j=1}^{n} j\, \vec{u}_j$, $\vec{v} = \sum_{k=1}^{m} k\, \vec{v}_k$

$$|\vec{v}|^2 = \sum_{k=1}^{m} k^2 v^2 + \sum_{k=1}^{m-1} k(k+1)(-v^2) \leftarrow 120^\circ$$

$$= v^2\left[\sum_{k=1}^{m-1}(k^2 - k^2 - k) + m^2\right] = v^2\left[m^2 - \sum_{k=1}^{m-1} k\right] = \frac{m}{2}(m+1)v^2$$

Similarly $|u|^2 = n(n+1)v^2$

Since $\vec{u}$ and $\vec{v}$ lineanly indepedent, hence

$$0 < |u|^2|v|^2 - (u\cdot v)^2 \quad \text{Cauchy inequlity}$$

$$= \frac{1}{2}n(n+1)m(m+1)v^4 - (mn)^2(\vec{u}_n\cdot\vec{v}_m)^2 \quad \text{only } \vec{u}_n, \vec{v}_m \text{ are non-orthogonal}$$

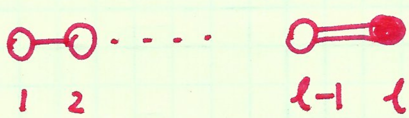
$$\downarrow 2v^2\cdot\cos^2 135^\circ = v^2$$

$$= \frac{1}{2}nm\left[(n+1)(m+1) - 2mn\right]v^2$$

$$= \frac{1}{2}nm\left[n+m+1-mn\right]v^2$$

Hence $n+m+1-mn > 0 \Rightarrow \boxed{(m-1)(n-1) < 2}$

① $m=1$, n arbitary



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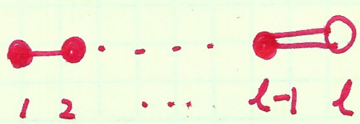
② $n=1$, m arbitary



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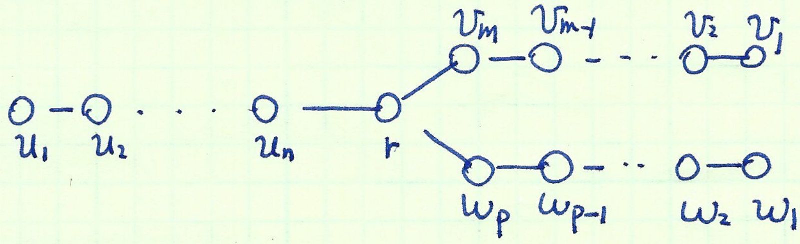
③ $m=n=2$

(F_4)

(*) Dynkin diagram only with single lines

① Diagrams with a fork — one root connecting three roots



All simple roots have equal length, denoted as "v"

Define $\vec{u} = \sum_{j=1}^{n} j\,\vec{u}_j \qquad \vec{v} = \sum_{k=1}^{m} k\,\vec{v}_k, \qquad \vec{w} = \sum_{s=1}^{p} s\,\vec{w}_s$

They are orthogonal to each other, and linearly indepedent with $\vec{r}$

$$v^2 = |r|^2 > \frac{(\vec{r}\cdot\vec{u})^2}{|u|^2} + \frac{(\vec{r}\cdot\vec{v})^2}{|v|^2} + \frac{(\vec{r}\cdot\vec{w})^2}{|w|^2}$$

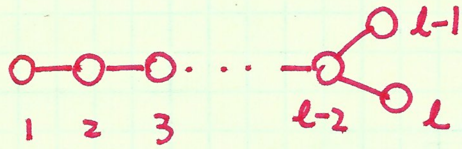
Accordig to the previous page,

$$\frac{(\vec{r}\cdot\vec{u})^2}{|u|^2} = \frac{(\vec{r}\cdot\vec{u}_n)^2}{\frac{1}{2}n(n+1)\,v^2} = \frac{n^2v^2/4}{\frac{1}{2}n(n+1)\,v^2} = \frac{v^2}{2}\left[1-\frac{1}{n+1}\right]$$

hence $\frac{3}{2} - \frac{1}{2}\left[\frac{1}{n+1} + \frac{1}{m+1} + \frac{1}{p+1}\right] < 1$

or $\frac{1}{n+1} + \frac{1}{m+1} + \frac{1}{p+1} > 1$

Assume $p \le m \le n$, we have ① $p = m = 1$, n arbitany



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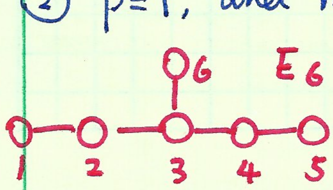
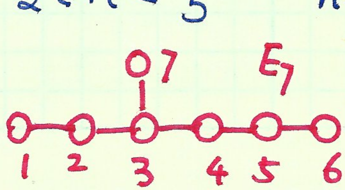
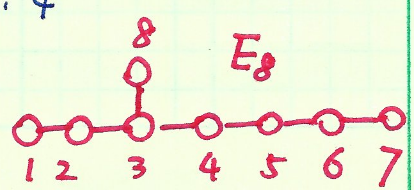
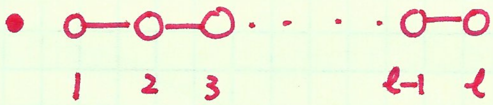
② $p=1$, and $m=2 \Rightarrow 2 \le n < 5$ $\quad n=2,3,4$

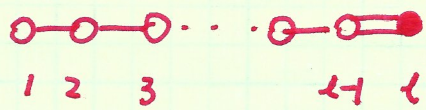


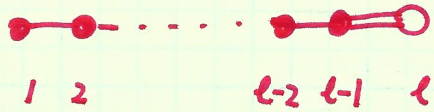


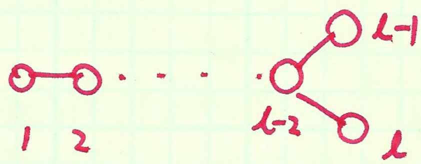


The last one

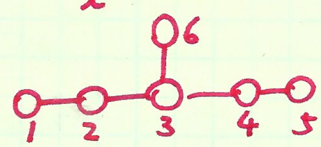
A_l $(l \geq 1)$  $SU(l+1)$

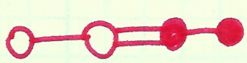
B_l $(l \geq 3)$  $SO(2l+1)$

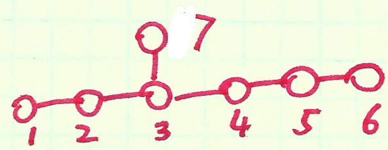
C_l $(l \geq 2)$  $Sp(2l)$

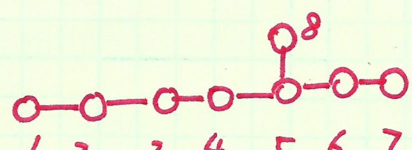
D_l $l \geq 4$  $SO(2l)$

G_2 

E_6 

F_4 

E_7 

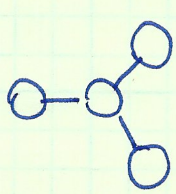
E_8 

$SU(2) \sim SO(3) \sim Sp(2)$ 

$SU(4) \sim SO(6)$ 

several equivalence equation

$Sp(4) \sim SO(5)$ 

$SO(8)$ 

triality