Lect 8 Young Pattern and Young tableau for SU(N) ①

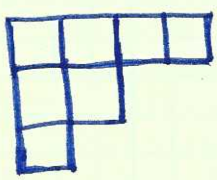
§ Fundementals of the permutatiun group

A permutatiun of n-object can be decamposed into a set of rotatiuns without commun elements

$$\underbrace{(a_1, \cdots a_{\lambda_1})}_{\lambda_1} \; \underbrace{(a_{\lambda_1+1}, \cdots a_{\lambda_1+\lambda_2})}_{\lambda_2} \cdots \underbrace{(a_{n-\lambda_n+1}, \cdots a_n)}_{\lambda_n}.$$

we want $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = n$.

The class ~~structure~~ is determined by this partitiun, denoted as

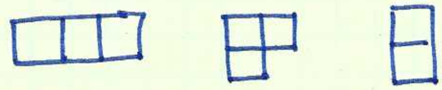


λ_1
λ_2
λ_3

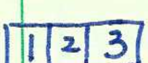
This is called Young pattern.

Each young pattern $\longleftrightarrow$ one class

\# of Young patterns $\longleftrightarrow$ \# of classes, \# of representatiuns

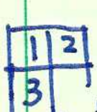
For example: for S_3  3 - representatiun / 3 - classes

Young tableau: Fill in young pattern; in each row, the \# on the right > \# on the left, and in each column, \# in the bottom > \# on the top. \# of Young tableau = dimesiun of Rep.



- dim = 1

dim = 1



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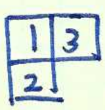


dim = 2

The dimension of Reps can be calculated by $d_{[\lambda]} = \dfrac{n!}{\prod_{ij} h_{ij}}$

h_{ij} is the hook number : $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \Rightarrow d_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, \\ \cline{1-1}\end{array}} = \dfrac{3!}{3\cdot 1\cdot 1} = 2.$

§ Young operator

For a young tableau, the permutation of numbers of each row is sumed to P_j, and $P = \prod P_j$. The permutation of numbers in each column multiplied by its parity summed to Q_i and $Q = \prod Q_i$, then $y = PQ$. For example

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad y = E + (12) + (23) + (13) + (123) + (132)$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad y = E - (12) - (23) - (13) + (123) + (132)$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad y = (E + (12))(E - (13)) = E + (12) - (13) - (132)$ (13)

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad y = (E + (13))(E - (12)) = E - (12) + (13) - (123)$

⊛ Young operator satisfies $y^2 = \lambda y$ with $\lambda = \dfrac{n!}{d}$ where d is the dimension of the representation of the associated Young pattern. For example $\left(y_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}\right)^2 =$

$$
\begin{aligned}
& E + (12) - (13) - (132) \\
& + (12) + E - \cancel{(123)} - \cancel{(23)} \\
3\left[E + (12) - (13) - (132)\right] \Rightarrow \quad & - (13) - (132) + E + (12) \\
& - (132) - \cancel{(13)} + \cancel{(23)} + \cancel{(123)}
\end{aligned}
$$

Hence Y is essentially the projection operater

$$e = \frac{1}{\lambda} Y \quad \text{satisfys} \quad e^2 = e.$$

we can use the Y-operater to project to each invariant space. For example, the group algebra S_3 is 6-dimensional. It can be decomposed as

$$\boxed{L = Le^{[3]} + Le_1^{[2,1]} + Le_2^{[2,1]} + Le^{[111]}}$$

L: group algebra of S_3.

HW For S_3: all of $e^{[3]}$, $e_1^{[2,1]}$, $e_2^{[2,1]}$, $e^{[111]}$ are orthogonal to each other, and they are complete. ~~For~~ Applying any group operation to Le, it is an invariant space

For general S_n, group. e,s for different Young patterns are othogonal. But e's for different young tableau of the same young pattern could be non-orthogonal. In this case, more work is needed to ensure the construction of the orthogonal projection operaters "e". Now let's just focus on S_3.

Check

① $b^{[3]} = e^{[3]} = \frac{1}{6}(E + (12) + (23) + (13) + (123) + (132))$

any element leaves $b^{[3]}$ invariemt, hence, this is the identity Rep.

② $b^{[21]}_{11} = e^{[21]}_{1} = \frac{1}{3}[E + (12) - (13) - (213)]$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$

$b^{[21]}_{21} = (23)\, e^{[21]}_{1} = \frac{1}{3}[(23) + (132) - (123) - (21)]$

$(12)\, b^{[21]}_{11} = \frac{1}{3}[(12) + E - (132) - (13)] = b^{[21]}_{11}$

$(12)\, b^{[21]}_{21} = \frac{1}{3}(\,(123) + (13) - (23)\,\underset{-E}{]} = -\,b^{[21]}_{11} - b^{[21]}_{21}$

$\Rightarrow$ the matrix for (12): $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

For (123)

$(123)\, b^{[21]}_{11} = \frac{1}{3}[(123) + (13) - (23) - \overset{E}{}] = -\,b^{[21]}_{11} - b^{[21]}_{21}$

$(123)\, b^{[21]}_{21} = \frac{1}{3}(\,(132) + (23) - (12) - (123)] = b^{[21]}_{21}$

$\Rightarrow$ (123): $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

③ The representation based on $e^{[21]}_{2}$ gives rise to the same Rep

④ $b^{[111]} = \frac{1}{6}(E - (12) - (23) - (13) + (123) + (132)]$

$\Rightarrow$ the staggered Rep.

§ Tensor for SU(3) group

Consider a 3D complex space, whose vector is a 3-d $\widehat{\text{complex}}$ vector.

$$V_a \xrightarrow{u} V_a' \equiv (O_u V)_a = u_{ab} V_b \qquad a = 1, 2, 3$$

u is a 3×3 SU(3) matrix

The rank-n tensor has n-indices, 3^n components,

$$T_{a_1 a_2 \cdots a_n} \xrightarrow{u} (O_u T)_{a_1, \cdots a_n} = \sum_{b_1 \cdots b_n} u_{a_1 b_1} \cdots u_{a_n b_n} T_{b_1 \cdots b_n}.$$

The rank-n tensor span a 3^n dimensional tensor space, which is also invariant under the SU(3) transformation. The n-th tensor components form n-objects for the permutation S_n group. The tensor space is invariant under permutation group S_n. Let me define the rule of the permutation R to the tensor index

$$R = \begin{pmatrix} 1 & 2 & \cdots & n \\ r_1 & \cdots & & r_n \end{pmatrix} = \begin{pmatrix} s_1 & \cdots & s_n \\ 1 & \cdots & n \end{pmatrix}.$$

We define the action of R to $T_{a_1 \cdots a_n}$ is to change the j-th index a_j to the r_j-th position $\Leftrightarrow$ the j-th position is filled with the original a_{s_j} position. For example $R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Then $R\, T_{\underline{a_1}\underline{a_2}\underline{a_3}} = T_{a_3 a_1 a_2}$ (move the object on the j-th position permutation on positions)

or more generally $\boxed{R\, T_{a_1 a_2 \cdots a_n} = T_{s_1 \cdots s_n}}$

(*) The permutation among the tensor indices and the SU(3) group operation O_u are commutable. — They work on different spaces.

Example: $(12)\, T_{a_1a_2} = T_{a_2a_1}$

$$O_u(12)\, T_{a_1a_2} = O_u T_{a_2a_1} = u_{a_2b_2} u_{a_1b_1} T_{b_2b_1}$$

$$(12)\left[O_u T\right]_{a_1a_2} = (12)\, u_{a_1b_1} u_{a_2b_2} T_{b_1b_2} = u_{a_1b_1} u_{a_2b_2} T_{b_2b_1}$$

HW: To prove the general case $O_u R\, T_{a_1\cdots a_n} = R\, O_u T_{a_1\cdots a_n}$

Hence, the permutation symmetry of the tensor index is kept when in the SU(3) transformation. For example. consider the rank-2 tensor $T_{ab} = \frac{1}{2}\{T_{ab} + T_{ba}\} + \frac{1}{2}\{T_{ab} - T_{ba}\}$

or $T_{ab} = \frac{1}{2}\{E + (12)\}T_{ab} + \frac{1}{2}(E - (12))T_{ab}$

$$= \frac{1}{2}\{y^{[2]} + y^{[1,1]}\}\, T_{ab} = E\, T_{ab}$$

Hence, we use the Young operator as projection operator, to project the tensor space into an invariant subspace for the SU(3) transformation. For example, the rank-3 tensor can be

$$T_{abc} = \tfrac{1}{6} Y^{[3]} T_{abc} + \tfrac{1}{3} Y_1^{[2,1]} T_{abc} + \tfrac{1}{3} Y_2^{[2,1]} T_{abc} + \tfrac{1}{6} Y^{[111]} T_{abc}$$

The rank-3 tensor space is decomposed into 4 invariant subspaces. The two in the middle are mixed tensors

$$y^{[3]}T_{abc} = \{E + (23) + (13) + (12) + (123) + (321)\}\, T_{abc}$$

$$= T_{abc} + T_{acb} + T_{cba} + T_{bac} + T_{bca} + T_{cab}$$

$$y_1^{[21]}T_{abc} = \{E + (12) - (13) - (213)\}\, T_{abc}$$

$$= T_{abc} + T_{bac} - T_{cba} - T_{bca}$$

$$y_2^{[21]}T_{abc} = \{E + (13) - (12) - (312)\}\, T_{abc} = T_{abc} + T_{cba} - T_{bac} - T_{cab}$$

$$y^{[111]}T_{abc} = T_{abc} - T_{acb} - T_{cba} - T_{bac} + T_{bca} + T_{cab}$$

what's the dimension in each sector? Each of them forms an irreducible Rep of the SU(3) group. The total dimension $3^3 = 27$.

dimension of SU(N)

$$\boxed{\ }\boxed{\ }\boxed{\ } = \frac{\boxed{N}\boxed{N+1}\boxed{N+2}}{\boxed{3}\boxed{2}\boxed{1}} = \frac{1}{6}N(N+1)(N+2) \rightarrow 10 \text{ dim for } SU(3)$$

$$\dim\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \frac{\begin{array}{|c|c|}\hline N & N+1 \\ \hline N-1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = \frac{N(N+1)(N-1)}{3} \rightarrow 8 \text{ dim for } SU(3)$$

$$\dim\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right) = \frac{\begin{array}{|c|}\hline N \\ \hline N-1 \\ \hline N-2 \\ \hline\end{array}}{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} = \frac{1}{6}N(N-1)(N-2) \rightarrow 1 \text{ for } SU(3)$$

$$\boxed{10 + 8 \times 2 + 1 = 27}$$

General rule for Represntation of SU(N). —

Each irredicible representation is denoted by a Young pattern, its # of rows less or equal to N−1. The dimension of Representation is a ratio of two #s.

The numertir is the product of numbers filling in the Young pattern following the rule of .

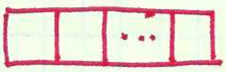
N	N+1	N+2	N+3
N-1	N	N+1	
N-2	N-1		
N-3			

The denorminatur is the hook #

7	5	3	1
5	3	1	
3	1		
1			

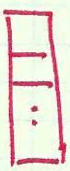
HW: According to the above rule, figure out the dimensins of the following Reps of the SU(N) group.

① one row fully symmetric Rep

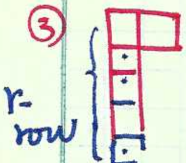


one row r-boxes $\rightarrow$ $\dim = \binom{N+r-1}{r}$

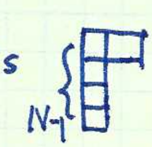
② one column fully anti-symmetric Rep



one column r-boxes $\dim = \binom{N}{r}$

③ 

$\dim = \frac{(N+1)N}{r+1}\binom{N-1}{r-1}$

The adjoing Rep is 

$\Rightarrow \dim = N+1\binom{N-1}{N-2} = N^2-1$

⊗ Construction of basis of irreducible tensor ~~represen~~ space

Let us consider the tensor sub space projected by the Young tableau

$$Y^{[2,1]}_1 \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} : \left[E + (12) - (13) - (132)\right]$$

projection on Tabc

$Y^{[21]}_1 T_{abc}$ is denote as $\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, i.e, put the ith index of T input the position denoted by the number "i" in the Young tablea

$$\text{The } \begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = Y^{[21]}_1 T_{abc} = T_{abc} + T_{bac} - T_{cba} - T_{bca}$$

Now we fill in 1 2 3 for a, b, c in this Young tableau

① we cannot put a b c with the same numbers.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2T_{112} - T_{211} - T_{121}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2T_{113} - T_{311} - T_{131}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2T_{223} - T_{322} - T_{232}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_{122} + T_{212} - 2T_{221}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_{133} + T_{313} - 2T_{331}$$

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_{233} + T_{323} - 2T_{332}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_{123} + T_{213} - T_{321} - T_{2}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_{132} + T_{312} - T_{231} - T_{3}$$

These are 8-linearly independ basis. Other fillings do not yield new basis. Relations

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = - \begin{array}{|c|c|}\hline c & b \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline b & a \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline a & c \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 0$$

HW: work out the bases for the other three tensor subspaces based on ① The fully symmetric tensor subspace projected by $Y^{[3]}$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$, ② another mixed tensor subspace $Y^{[21]}_{2}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, ③ the fully anti-symmetric one $Y^{[1,1,1]}$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$. (Rank-3 tensors for SU(3) group

Basically, the idea of using Young tableau + Young operator to decompose the tensor subspace is summarized as.

$$
\begin{array}{lll}
 & \xrightarrow{\hspace{6em}} \text{SU(N) Rep.} & \\
Y^{[\lambda]}_{\mu_1} T_{a_1 b_1 c_1} & Y^{[\lambda]}_{\mu_1} T_{a_2 b_2 c_2} & \cdots\cdots \\
Y^{[\lambda]}_{\mu_2} T_{a_1 b_1 c_1} & Y^{[\lambda]}_{\mu_2} T_{a_2 b_2 c_2} & \cdots \\
\vdots & &
\end{array}
$$

S_n Rep $\big\downarrow$

Different young tableaus (based on the same Young pattern) project out equivalent tensor subspaces: $T^{[\lambda]}_{\mu}$ and $T^{[\lambda]}_{\nu}$. Their Young operators $Y^{[\lambda]}_{\mu}$ and $Y^{[\lambda]}_{\nu}$ are connected by permutat $R_{\mu\nu}$, i.e $Y^{[\lambda]}_{\nu} = R_{\mu\nu} Y^{[\lambda]}_{\mu} R^{-1}_{\mu\nu}$. Hence the tensor ~~bases~~ represen-tations are equivalent for SU(N).

Actually, these equivalent subspaces form ~~a~~ representations for the permutation group S_n, where n is the rank of tensors.