Gaussian model and Ginzburg crieterium

Let's only keep the quadratic term of the Landau free energy at $T > T_c$.

$$Z = \int Dm \; e^{-F(m)}, \quad \text{where } F(m) = \int d^d x \, \frac{\gamma}{2} (\nabla m)^2 + \frac{\alpha(T)}{2} m^2 - m(x) h(x).$$

$$F(m) = \sum_k m(k) m(-k) \left[\frac{\gamma}{2} k^2 + \frac{\alpha(T)}{2} \right] - m(k) h(-k)$$

$$= \sum_k \left[m(k) - \frac{h(k)}{\gamma k^2 + \alpha} \right] \left[m(-k) - \frac{h(-k)}{\gamma k^2 + \alpha} \right] \left[\frac{\gamma}{2} k^2 + \frac{\alpha(T)}{2} \right]$$

$$- \sum_k \frac{1}{2} \frac{h(k) h(-k)}{\gamma k^2 + \alpha}$$

denote $m'(k) = m(k) - \dfrac{h(k)}{\gamma k^2 + \alpha}$. Because m and h are real field, $\Rightarrow m(k) = m^*(-k)$. $\Rightarrow$

$$\sum_k m'(k) m'(-k) \frac{\gamma k^2 + \alpha}{2} = \sum_k{}' m^*(k) m(k) (\gamma k^2 + \alpha)$$

$$= \sum_k{}' [\mathrm{Re}\, m(k)]^2 (\gamma k^2 + \alpha) + \sum_k{}' \mathrm{Im}\, m(k)^2 (\gamma k^2 + \alpha)$$

$\sum'$ means summation over half of momentum space.

The Gaussian integral

$$\int Dm \; e^{-F(m,h)} = e^{-\sum_k \frac{1}{2} \frac{h(k) h(-k)}{\gamma k^2 + \alpha}} \prod_k{}' \int d\mathrm{Re}\, m'(k) \int d\mathrm{Im}\, m'(k) \; e^{-(\mathrm{Re}(m'(k)))^2 (\gamma k^2 + \alpha)} \, e^{-\mathrm{Im}(m'(k))^2 (\gamma k^2 + \alpha)}$$

$$= e^{-\sum_k \frac{1}{2} h(k) h(-k) \frac{1}{\gamma k^2 + \alpha}} \prod_k{}' \sqrt{\frac{\pi}{\gamma k^2 + \alpha}} \sqrt{\frac{\pi}{\gamma k^2 + \alpha}}$$

$$= \text{const.} \; e^{-\sum_k \frac{1}{2} h(k) h(-k) \frac{1}{\gamma k^2 + \alpha}} \prod_k \sqrt{\frac{\pi}{\gamma k^2 + \alpha}}$$

$$\Rightarrow \ln Z = -\frac{1}{2} \sum_k \ln(\gamma k^2 + \alpha) - \frac{1}{2} \sum_k \frac{h^*(k) h(k)}{\gamma k^2 + \alpha}$$

Set $h=0$, $\frac{F}{V} = -\frac{1}{V\beta} \ln Z = \frac{k_B T}{2} \int \frac{dk^d}{(2\pi)^d} \ln[\gamma k^2 + \alpha]$

since $\alpha \propto t \Rightarrow \frac{C}{V} = -T \frac{\partial^2 F}{\partial T} \propto \int_0^{\Lambda} \frac{dk^d}{(2\pi)^d} \frac{1}{(\gamma k^2 + \alpha)^2}$ Λ is the momentum space cut off.

This integral has two possible divergence $\begin{cases} ① \ \Lambda \to \infty \text{ if so, ultra-violet} \\ ② \ \alpha \to 0 \text{ if so infra red.} \end{cases}$

① if $d>4$, at $\alpha \propto t \to 0$, there's no infrared divergence.

we have ultra-violet divergence, but this divergence is not related to phase transition.

② if $d<4$, as $t \to 0$, there's infrared divergence.

But as $\Lambda \to \infty$. no divergence.

③ if $d=4$, we have logrithmic divergence as both $t \to 0$

and $\Lambda \to \infty$.

check case ②

$$\int_0^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(\gamma k^2 + \alpha)^2} = \int_0^{\Lambda} \frac{dk}{(2\pi)^d} \frac{k^{d-1}}{\alpha^2 \left[\left(\frac{k}{\sqrt{\alpha/\gamma}}\right)^2 + 1\right]} \propto t^{\frac{d-4}{2}} \int_0^{\Lambda'} ds \frac{s^{d-1}}{(s^2+1)^2}$$

with $\left. \begin{array}{l} \Lambda' = \dfrac{\Lambda}{\sqrt{\frac{\alpha_0}{\gamma} t}} \\ \text{and } \alpha = \alpha_0 t \end{array} \right\}$

$\Rightarrow$ at $d<4$, we can extra the infrred divergence

$$\frac{C}{V} \propto \text{Cnst.}\ t^{\frac{d-4}{2}} \Rightarrow \boxed{\alpha = \frac{4-d}{2}}$$

check case ①: as $t \to 0$, $\frac{C}{V} \propto \int_0^\Lambda \frac{d^d k}{(2\pi)} \frac{1}{\gamma^2 k^4} \propto \Lambda$-dependent const.

check case ③: $\frac{C}{V} \propto \int_0^\Lambda \frac{k^3 dk}{(k^2+\alpha/\gamma)^2} \propto \int_0^\Lambda \frac{k^2 dk^2}{(k^2+\alpha/\gamma)^2}$ set $y = k^2 + \alpha/\gamma$

$$\Rightarrow \frac{C}{V} \propto \int_{\alpha/\gamma}^{\Lambda^2} \frac{(y-\alpha/\gamma)\,dy}{y^2} \sim \ln\frac{\Lambda^2}{\alpha/\gamma} \sim \text{ultraviolet const} + \ln\frac{1}{t}.$$

The above result shows that we cannot neglect the Gaussian fluctuation at $d<4$, and we also need to be careful at $d=4$. At $d>4$, the ultra-violet divergence does not affect the phase transition.

⊛ Ginzburg criterion

The GL theory (mean field) fails when the fluctuations is strong.

Define $\frac{1}{V}\int_{\xi} dr\ m(r)$. The integral over the size of correlation length ξ.

The long range order $\bar{m} = \frac{1}{V}\langle \int dr\ m(r) \rangle = \frac{\alpha_0}{\beta}|t|$

$$\text{define } \frac{\frac{1}{\xi^d}\int d^d r\left\{\langle m(0)\, m(r)\rangle - \bar{m}^2\right\}}{\bar{m}^2} = \frac{1/\xi^d \int dr\ G(r)}{\bar{m}^2} = E_{GL}$$

which is a characteristic quantity to judge the fluctuation effect

The denominator :

$$\begin{cases} \bar{m}^2 = \dfrac{-\alpha(T)}{\beta} = \dfrac{\alpha_0}{\beta}|t| \\ \xi^2(t) = \dfrac{\gamma}{\alpha_0}|t|^{-1} = \xi^2(1)\,|t|^{-1} \end{cases}$$

where $\xi(1)$ is the correlation length far away free the critical region. ($\xi(1) = \sqrt{\frac{\gamma}{\alpha_0}}$).

$$\Rightarrow \boxed{\text{denominator } \ \frac{\alpha_0}{\beta}\,\xi^d(1)\,|t|^{1-d/2}}$$

The numerator

$$\int dr\ G(r) \simeq k_B T\,\chi_T \simeq k_B T_c \frac{1}{4\alpha_0|t|}$$

$$E_{GL} = \frac{k_B T_c}{4\alpha_0 |t|} \frac{\beta}{\alpha_0 \xi^d(1) |t|^{1-d/2}} = \frac{k_B}{4\Delta C \xi^d(1)} \frac{1}{|t|^{2-d/2}} = \frac{k_B |t|^{d/2-2}}{4\Delta C \xi^d(1)}$$

where $\Delta C = \frac{\alpha_0^2}{\beta} T_c$ is the mean field specific heat jump at the transition. If $E_{GL} \ll 1$, then the GL theory is self consistent, otherwise, the GL theory breaks down and we enters the critical fluctuation regime.

① At $d > 4$, $E_{GL} \sim |t|^{d/2-2} \ll 1$ as $t \to 0$.
Landau - Ginsburg theory are qualitatively correct.

② At $d < 4$. $E_{GL} \sim |t|^{d/2-2} \gg 1$ as $t \to 0$. The mean-field theory breaks down at $E_{GL} \simeq 1$, i.e. $|t|^{-d/2+2} = \frac{k_B}{4\Delta C \xi^d(1)}$

i.e. at $|t| < |t_c| = \left[\frac{k_B}{4\Delta C \xi^d(1)}\right]^{\frac{1}{2-d/2}}$,

we enter the critical region.

③ $d = 4$ is the marginal case.

There exists a $\boxed{\text{upper critical dimension } d_c = 4}$ for the above analysis, such at at $d > d_c$, the quartic term is not important for the critical phenomenon. Of course, we need quartic term to spontaneously break the symmetry!

i.e. interaction

☆ In the above reasoning, we have use the mean field values of critical exponents $\beta = 1/2$, $\gamma = 1$, $\nu = 1/2$. However for certain mean field transitions whose γ, β, ν have different values, we need modify as follows:

$$\int_V d^d r \, G(r) \sim k_B T \chi_T \sim |t|^{-\gamma}$$

$$\xi^d \bar{m}^2 \sim \xi^d |t|^{2\beta} \sim |t|^{2\beta - \nu d}$$

Suppressing numerical coefficient, we need $|t|^{-\gamma} << |t|^{2\beta - \nu d}$ to justify GL mean field theory: $-\gamma > 2\beta - \nu d$

$$\Rightarrow \boxed{d > \frac{2\beta + \gamma}{\nu} \triangleq d_c}$$