Solutions to 1D Ising models

1 1D Ising model

$$\begin{cases} H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \\ \sigma_i = \pm 1 \end{cases}$$

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H}$$

$\{\sigma_i\}$ refers to the distributions of σ_i, and $\beta = \frac{1}{k_B T}$

- Lenz proposed this model for ferromagnetism as the Ph.D. thesis topic for Ising. Ising solved the 1D case and found there's no phase transition. Onsager solved the 2D case, and indeed found the PM - FM transition. In 3D, there's no exact solution so far — a NP hard problem.

- Why there's no phase transition at 1D for any $T > 0$, always PM.

Consider low T limit, and we start from one of the ground state

$$\cdots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \cdots .$$

The lowest energy excitation is not flipping one spin, but a kink!

$$\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow \qquad \Delta E = 2J \quad (\text{topological} \leftrightarrow \textstyle\sum \sigma_i = 0)$$

$$\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow \qquad \Delta E = 4J \quad (\text{non-topological} \leftrightarrow \textstyle\sum \sigma_i \sim N)$$

Thus the kink is less expensive, and more efficient to destroy magnetic ordering. At $T \ll J$, we only count one of the ground state, and all the single kink states.

The location of the kink can be at any bond —— $N-1$ fold degeneracy.

The probility of kinks relative to $\uparrow\uparrow\uparrow\cdots\uparrow$, is $e^{-\beta 2J}\cdot(N-1)$, or

if we take logrithim $\rightarrow$ $\Delta F = 2J - \frac{1}{\beta}\ln(N-1) < 0$ (as $N\to\infty$).

No matter how small T is, the kink configuration domainates over the polarized configuration, which destroy the ordered state.

② Calculation of partition function of 1d Ising model

a: at $h=0$, and open boundary condition, we can simplify by defining bond variable $t_i = \sigma_i\sigma_{i+1}$, for $i=1,2,\cdots N-1$. Then each distribution of $\{t_i\}$ for $i=1,\cdots N-1$ corresponds to 2 sets value of $\{\sigma_i\}$, $-\{\sigma_i\}$. with an overall sign.

Then 1D Ising model became $N-1$ decoupled sites

$$Z = 2\left(e^{J/kT} + e^{-J/kT}\right)^{N-1} = \overset{N}{2}\left[\cosh\beta J\right]^{N-1}$$

$$\frac{F}{N} \xrightarrow[N\to\infty]{} -\frac{1}{\beta}\ln\left[2\cosh\beta J\right].$$

This is an analytical function of T, at $T>0$, thus there's no phase transition.

b: more generally, we introduce the method of transfer matrix.

$$Z = \sum_{\sigma_i=\pm1,\ i=1,\cdots N} e^{\beta J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + \frac{\beta h}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1})}$$

periodicall boundary condition

$$= \sum_{\{\sigma_i\}} T_{\sigma_1\sigma_2} T_{\sigma_2\sigma_3}\cdots T_{\sigma_N\sigma_1}$$

$$T_{\sigma_i \sigma_{i+1}} = e^{\beta J \sigma_i \sigma_{i+1} + \frac{\beta}{2} h(\sigma_i + \sigma_{i+1})} \xrightarrow[\text{as a matrix}]{} \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$Z = \operatorname{tr} T^N$, denote $\lambda_\pm$ as the eigenvalues of T

$$\Rightarrow \ Z = \lambda_+^N + \lambda_-^N \qquad (e^{\beta(J+h)} - \lambda)(e^{\beta(J-h)} - \lambda) - e^{-2\beta J} = 0$$

$$\lambda^2 - 2\cosh\beta h \, e^{\beta J} + e^{2\beta J} - e^{-2\beta J} = 0$$

$$\boxed{\lambda_\pm = e^{\beta J}\left(\cosh\beta h \pm \sqrt{\sinh^2\beta h + e^{-4\beta J}}\right)}$$

$$\frac{F}{N} = -\frac{1}{\beta N} \ln(\lambda_+^N + \lambda_-^N)$$

$$= -\frac{1}{\beta}\ln\lambda_+ - \frac{1}{\beta N}\ln\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

$\longrightarrow -\frac{1}{\beta}\ln\lambda_+$ as $N \to +\infty$. we only need the largest value of λ_+.

$$\frac{F}{N} = -\frac{1}{\beta}\ln\left(e^{\beta J}\cosh\beta h + e^{-\beta J}\sqrt{1 + e^{4\beta J}\sinh^2(\beta h)}\right)$$

at $h \to 0 \longrightarrow -\frac{1}{\beta}\ln 2\cosh\beta J$.

$$M = -\frac{\partial F}{\partial h} = \frac{1}{\beta} \frac{\beta\left[e^{\beta J}\sinh\beta h + e^{-\beta J}\frac{1}{2}(1 + e^{4\beta J}\sinh^2\beta h)^{-1/2} e^{4\beta J} 2\sinh\beta h \cosh\beta h\right]}{e^{\beta J}\cosh\beta h + e^{-\beta J}\sqrt{1 + e^{4\beta J}\sinh^2\beta h}}$$

$$= \frac{\sinh\beta h\left[1 + \frac{e^{2\beta J}}{\sqrt{1 + e^{4\beta J}\sinh^2\beta h}}\cosh\beta h\right]}{\cosh\beta h + e^{-2\beta J}\sqrt{1 + e^{4\beta J}\sinh^2\beta h}} \longrightarrow 0, \text{ at } h \to 0 \text{ first, and any } T.$$

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$$= \frac{\sinh\beta h + \frac{1}{\sqrt{e^{-4\beta J}\sinh^{-1}\beta h + 1}}\cosh\beta h}{\cosh\beta h + \sqrt{e^{-4\beta J} + \sinh^2\beta h}}$$

$\rightarrow \mathbf{1}$ if $\beta \to \infty$ first at any h.

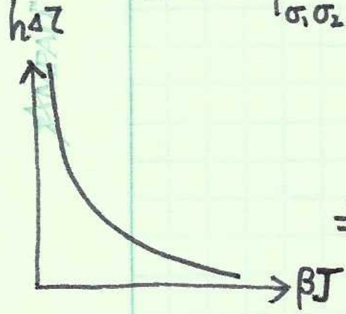
§ map to a quantum mechanical problem

$$Z = \sum_{\{\sigma\}} e^{\beta J \sum_i (\sigma_i \sigma_{i+1} - 1)}, \quad N \text{ sites.}$$

then we have $\quad Z = \sum_{\sigma_1, \cdots \sigma_N} T_{\sigma_1\sigma_2} T_{\sigma_2\sigma_3} \cdots T_{\sigma_N\sigma_1} = \mathrm{tr}\, T^N$

with $\quad T_{\sigma_1\sigma_2} = \begin{pmatrix} 1 & e^{-2\beta J} \\ e^{-2\beta J} & 1 \end{pmatrix} = 1 + e^{-2\beta J} \sigma_1 = \text{const}\; e^{h\Delta\tau\, \sigma_1}$

$$= \text{const}\left[\cosh \Delta\tau h + \sinh \Delta\tau h \; \sigma_1\right]$$



$$\Rightarrow \tanh h\Delta\tau = e^{-2\beta J} \qquad \Rightarrow \sinh 2\beta J = \frac{1}{2}\left[\frac{\mathrm{ch}\, h\Delta\tau}{\mathrm{sh}\, h\Delta\tau} - \frac{\mathrm{sh}\, h\Delta\tau}{\mathrm{ch}\, h\Delta\tau}\right] = \frac{1}{\sinh h\Delta\tau}$$

$$\Rightarrow \boxed{\sinh 2\beta J \; \sinh h\Delta\tau = 1}$$

(only the product $h\Delta\tau$ is well-defined).

$$\Rightarrow \quad Z = \mathrm{tr}\left[e^{+N\Delta\tau\, h\sigma_1}\right]$$

$$= \mathrm{tr}\left[e^{-\beta_\tau h \sigma_1}\right] \quad \text{where } \beta_\tau = N\Delta\tau \xrightarrow{N\to\infty} \infty.$$

Thus the $\underbrace{}_{\text{1D}}$ thermo partition function can be mapped into a quantum partition function of a single spin with a transverse field.

It's clear that there's no spin polarization along z-axis.

(No phase transition. 1D Ising cannot describe finite T at any finite T phase transition.)

§ spin-spin correlation function (1D)

Using the method of transfer matrix, we can also calculate the spin-spin correlation function.

$$Z = tr\left[\left(e^{h\Delta\tau\sigma_1}\right)^N\right] \quad \text{where } \sinh h\Delta\tau \, \sinh 2\beta J = 1.$$

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum \sigma_i \sigma_j \, e^{\beta J(\Sigma \sigma_m \sigma_{m+1} - 1)}}{\sum_{\{\sigma\}} e^{\beta J \Sigma(\sigma_m \sigma_{m+1} - 1)}} = \frac{tr\left[T^{N-i} \sigma_3 T^{i-j} \sigma_3 T^{j}\right]}{tr\, T^N}$$

assume $i > j$

$$= tr\left[T^N \, T^{-i} \sigma_3 T^{i} \, T^{-j} \sigma_3 T^{j}\right] / tr\left[T^N\right], \text{ where } T = e^{h\Delta\tau\sigma_1}.$$

define the eigenstates of $e^{h\Delta\tau\sigma_1}$ as $|0\rangle$, and $|1\rangle$ with $\lambda = e^{h\Delta\tau}$ and $e^{-h\Delta\tau}$ respectively, i.e $T = e^{h\Delta\tau} |0\rangle\langle 0| + e^{-h\Delta\tau} |1\rangle\langle 1|$.

as $N \to \infty$, $T^N = e^{Nh\Delta\tau} |0\rangle\langle 0|$.

$$\Rightarrow \langle \sigma_i \sigma_j \rangle \xrightarrow[\text{assume } i > j]{} \langle 0| T^N |0\rangle \langle 0| T^{-i} \sigma_3 T^{i-j} \sigma_3 T^{j} |0\rangle / \langle 0| T^N |0\rangle$$

$\boxed{\text{T is the imaginary-time evolution operator}}$

$$= \langle 0| T^{-i} \sigma_3 T^{i-j} \sigma_3 T^{j} |0\rangle$$

$$= \langle 0| T^{-i} |0\rangle \langle 0| \sigma_3 |a\rangle \langle a| T^{i-j} |a\rangle \langle a| \sigma_3 |0\rangle \langle 0| T^{j} |0\rangle$$

$|0\rangle$ and $|1\rangle$ are σ_1 eigenstates, thus $\langle 0|\sigma_3|0\rangle = \langle 1|\sigma_3|1\rangle = 0$

$$\langle 0|\sigma_3|1\rangle = \langle 1|\sigma_3|0\rangle = 1$$

$$\Rightarrow \langle \sigma_i \sigma_j \rangle \underset{i > j}{=} \lambda^{-i} \langle 0|\sigma_3|1\rangle \left(\lambda^{-1}\right)^{(i-j)} \langle 1|\sigma_3|0\rangle \, \lambda^{+j}$$

$$= \lambda^{-2(i-j)} \left|\langle 0|\sigma_3|1\rangle\right|^2 = \left[e^{h\Delta\tau}\right]^{-2(i-j)} = e^{-|i-j|/\xi}$$

$$\Rightarrow \xi = \frac{1}{2h\Delta\tau} = \left[\ln \coth \beta J\right]^{-1} \leftarrow \left(e^{2h\Delta\tau} = \coth \beta J\right).$$