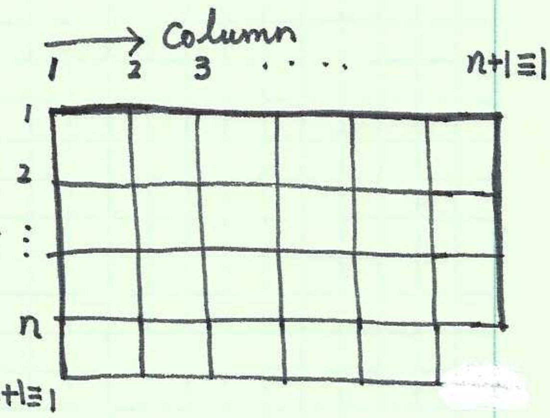
§ The onsager solution to 2D Ising model

1. Set up the representation.



n row $\times$ n column : $N = n^2$

use $\mu_\alpha \equiv \{\sigma_1, \sigma_2, \cdots \sigma_n\}$ - αth row

$\alpha = 1, 2, \cdots n$. μ_α represents the anfiguration of the α-th row.

The α-th row only interact with the $\alpha-1$th row and the $\alpha+1$ th row.

Use $E(\mu_\alpha, \mu_{\alpha+1})$ represent the coupling between α and $\alpha+1$th row, and $E(\mu_\alpha)$ to represent the coupling with in the α-th row.

$$E(\mu, \mu') = -J \sum_{k=1}^{n} \sigma_k \sigma_k'$$

$$E(\mu) = -J \sum_{k=1}^{n} \sigma_k \sigma_{k+1} - h \sum_{k=1}^{n} \sigma_k$$

(μ, μ' represent anfigurations of two adjacent rows. S_k, S_k' are spin within these two rows).

$$\mu = \{\sigma_1, \sigma_2 \cdots \sigma_n\}$$
$$\mu' = \{\sigma_1', \sigma_2', \cdots \sigma_n'\}$$

The $E\{\mu_1, \mu_2, \cdots \mu_n\} = \sum_{\alpha=1}^{n} \left[E\{\mu_\alpha, \mu_{\alpha+1}\} + E\{\mu_\alpha\} \right]$

and $Z[h, \beta] = \sum_{\mu_1} \cdots \sum_{\mu_n} \exp\left[-\beta \left\{ \sum_{\alpha=1}^{n} E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha) \right\} \right]$

Similarly to the method in the 1D case, we introduce the transfer matrix, but now this matrix is huge $2^n \times 2^n$ dimensional. Define

$$\langle \mu | P | \mu' \rangle \equiv e^{-\beta\left[E(\mu, \mu') + E(\mu)\right]}$$

Then $$Z[h,\beta] = \sum_{\mu_1} \cdots \sum_{\mu_n} \langle \mu_1 | P | \mu_2 \rangle \langle \mu_2 | P | \mu_3 \rangle \cdots \langle \mu_n | P | \mu_1 \rangle$$

$$= \sum_{\mu_1} \langle \mu_1 | P^n | \mu_1 \rangle = \mathrm{Tr}\, P^n$$

If we can diagonalize P as $\mathrm{diag}\{\lambda_1, \cdots \lambda_{2^n}\}$, and then

$$Z[h,\beta] = \sum_{\alpha=1}^{2^n} (\lambda_\alpha)^n$$

Since $E\{\mu, \mu'\}$ and $E\{\mu\}$ are at the order of n, thus λ's are at the order of e^n. For the largest ~~value~~ eigenvalue of P, defined as λ_{max}, we expect

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_{max} = \text{finite}.$$

If this is true and all the λ's are positive, then

$$\lambda_{max}^n \leq Z \leq 2^n (\lambda_{max})^n$$

$$\Rightarrow \frac{1}{n} \log \lambda_{max} \leq \frac{1}{n^2} \ln Z \leq \frac{1}{n} \log \lambda_{max} + \frac{1}{n} \ln 2$$

$$\Rightarrow \lim_{N\to\infty} \frac{1}{N} \ln Z = \lim_{n\to\infty} \frac{1}{n} \log \lambda_{max}.$$

$(N = n^2)$

Thus our next job is to find the largest eigenvalues of P.

§ The matrix P

the matrix elements of P is $\langle \sigma_1, \cdots \sigma_n | P | \sigma_1' \cdots \sigma_n' \rangle = \prod_{k=1}^{n} e^{\beta h \sigma_k} e^{\beta J \sigma_k \sigma_{k+1}} \cdot e^{\beta J \sigma_k \sigma_k'}$

Now we decompose P as a product of three matrices

$$\langle \sigma_1 \cdots \sigma_n | V_1' | \sigma_1' \cdots \sigma_n' \rangle \equiv \prod_{k=1}^{n} e^{\beta J \sigma_k \sigma_k'}$$

$$\langle \sigma_1 \cdots \sigma_n | V_2 | \sigma_1' \cdots \sigma_n' \rangle \equiv \delta_{\sigma_1 \sigma_1'} \cdots \delta_{\sigma_n \sigma_n'} \prod_{k=1}^{n} e^{\beta J \sigma_k \sigma_{k+1}}$$

$$\langle \sigma_1 \cdots \sigma_n | V_3 | \sigma_1' \cdots \sigma_n' \rangle \equiv \delta_{\sigma_1 \sigma_1'} \cdots \delta_{\sigma_n \sigma_n'} \prod_{k=1}^{n} e^{\beta h \sigma_k}$$

$$\Rightarrow \langle \sigma_1 \cdots \sigma_n | P | \sigma_1' \cdots \sigma_n' \rangle = \langle \sigma_1 \cdots | V_3 V_2 V_1' | \sigma_1' \cdots \sigma_n' \rangle .$$

(actually P itself is not hermitian, although V_3 V_2 and V_1' are.

Next we use Γ-matrix representation

V_1' is a direct product of n 2×2 identical matrices

$$V_1' = a \otimes a \otimes \cdots \otimes a , \quad \text{and} \quad \langle \sigma | a | \sigma' \rangle = e^{\beta J \sigma \sigma'} = \begin{bmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{bmatrix}$$

$$= e^{\beta J} + e^{-\beta J} \tau^1$$

we use the symbol of τ^1, τ^2, τ^3 represent Pauli matrices in order to avoid the confusion with σ index. Introducing angle $\tanh\theta = e^{-\beta J}/e^{\beta J} = e^{-2\beta J}$

$$\Rightarrow a = \sqrt{(e^{\beta J})^2 - (e^{-\beta J})^2} \; e^{\theta \tau^1} = \sqrt{2\sinh 2\beta J} \; e^{\theta \tau^1}$$

$$V' = [2\sinh 2\beta J]^{n/2} e^{\theta \tau_1^1} e^{\theta \tau_2^1} \cdots e^{\theta \tau_n^1} = [2\sinh 2\beta J]^{n/2} e^{\theta(\tau_1^1 + \cdots \tau_k^1)}$$

$$V_1 = \prod_{k=1}^{n} e^{\theta \tau_k^1} \quad \text{and} \quad \tanh\theta = e^{-2\beta J}$$

$$V_2 = \prod_{k=1}^{n} e^{\beta J \tau^3_k \tau^3_{k+1}} \quad , \text{ and } \quad V_3 = \prod_{k=1}^{n} e^{\beta h \tau^3_k}$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \quad P = \left[2 \sinh 2\beta \mathcal{E}\right]^{n/2} V_3 V_2 V_1$, where $V_3 = 1$ at $h = 0$.

§ Γ- matrices (math preparation)

Γ-matrix is a generalization of Pauli matrices. At level n, there are 2n+1 Γ-matrix anticommute with each other, and it dimension 2^n.

level $n=1$ $\quad \tau^1_1, \quad \tau^2_1, \quad \tau^3_1$

$n=2$ $\quad \tau^1_1 \otimes \tau^1_2, \; \tau^1_1 \otimes \tau^2_2, \; \tau^1_1 \otimes \tau^3_2, \quad \tau^2_1 \otimes 1_2, \; \tau^3_1 \otimes 1_2$

$n=3$ $\quad \tau^1_1 \otimes \tau^1_2 \otimes \tau^1_3, \; \tau^1_1 \otimes \tau^2_2 \otimes \tau^1_3, \; \tau^1_1 \otimes \tau^3_2 \otimes \tau^1_3, \; \tau^2_1 \otimes 1_2 \otimes \tau^1_3, \; \tau^3_1 \otimes 1_2 \otimes \tau^1_3$

$\quad \tau^2_1 \otimes 1_2 \otimes 1_3 \quad \tau^3_1 \otimes 1_2 \otimes 1_3$

⋮

using a convenient convention (reverse the sequence..)

$$\begin{cases} \Gamma_1 = \tau^3_1 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_n \\ \Gamma_2 = \tau^2_1 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_n \\ \Gamma_3 = \tau^1_1 \otimes \tau^3_2 \otimes 1_3 \cdots \otimes 1_n \\ \Gamma_4 = \tau^1_1 \otimes \tau^2_2 \otimes 1_3 \cdots \otimes 1_n \\ \Gamma_5 = \tau^1_1 \otimes \tau^1_2 \otimes \tau^3_3 \cdots \otimes 1_n \\ \Gamma_6 = \tau^1_1 \otimes \tau^1_2 \otimes \tau^2_1 \cdots \otimes 1_n \\ \vdots \\ \Gamma_{2n-1} = \tau^1_1 \otimes \tau^1_2 \otimes \tau^1_3 \cdots \otimes \tau^3_n \\ \Gamma_{2n} = \tau^1_1 \otimes \tau^1_2 \otimes \tau^1_3 \cdots \otimes \tau^2_n \end{cases}$$

$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$

Clifford algebra.

$\Gamma_{2n+1} = \tau^1_1 \otimes \tau^1_2 \otimes \cdots \otimes \tau^1_n$

$\Gamma^{\mu\nu} = i\Gamma^\mu \Gamma^\nu$ form fundamental Rep of $2n+1$ dimension $SO(2n+1)$.

if we only take the first $2n$, $\Gamma^{\mu\nu} = i\Gamma^\mu \Gamma^\nu$ form the Rep of $SO(2n)$

but it is reducible.

Define $$\Gamma'_\mu = \sum_{\nu=1}^{2n} \omega_{\mu\nu} \Gamma_\nu$$ and the matrix $\omega_{\mu\nu}$ satisfies

$\omega^T \omega = 1$, or $\sum_\mu \omega_{\mu\nu}\omega_{\mu\lambda} = \delta_{\nu\lambda}$

i.e. $$\begin{bmatrix} \Gamma'_1 \\ \Gamma'_2 \\ \vdots \\ \Gamma'_{2n} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1,2n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2,2n} \\ \vdots & & & \vdots \\ \omega_{2n,1} & \omega_{2n,2} & \cdots & \omega_{2n,2n} \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{2n} \end{bmatrix}$$ ω a rotation in $2n$-vector space.

This transformation is induced by a similar transformation of Γ

$$\Gamma'_\mu = S(\omega)\Gamma_\mu S^{-1}(\omega)$$ $S(\omega)$ is a rotation on spinor of 2^n-dimension

c.f. $$\sigma'_\mu = R^{-1}(g)\,\sigma_\mu\, R(g) = R_{\mu\nu}(g)\,\sigma_\nu$$

rotation operator in the spinor Rep

$$\begin{cases} R(g) = e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\theta} \\ g = g(\hat{n},\theta). \end{cases}$$

$$\Rightarrow \boxed{S(\omega)\Gamma_\mu S^{-1}(\omega) = \sum_{\nu=1}^{2n} \omega_{\mu\nu}\Gamma_\nu}$$

Define a rotation in the $\mu\nu$ plane, and this rotation is denote as

$$\omega(\mu\nu|\theta) = \begin{array}{c@{}c} \begin{array}{ccccccc} & & \mu & & \nu & & \end{array} & \\ \begin{bmatrix} & & \vdots & & \vdots & & \\ \cdots & & \cos\theta & \cdots & \sin\theta & \cdots & \\ \cdots & & -\sin\theta & \cdots & \cos\theta & \cdots & \\ & & \vdots & & \vdots & & \end{bmatrix} & \begin{array}{l} \\ \mu\text{th row} \\ \nu\text{th row} \\ \\ \end{array} \end{array}$$

lemma 1: for $\omega(\mu\nu:\theta)$, the correspondence $S^{\mu\nu}(\theta)$ is

$$\boxed{S^{\mu\nu}(\theta) = \exp[-\frac{\Gamma_\mu \Gamma_\nu}{2}\theta]}$$

$\mu\nu$ is not the indices of matrix element

$$(\Gamma_\mu \Gamma_\nu)^2 = -1$$

Proof:

$$S^{\mu\nu}(\theta) = \cos\frac{\theta}{2} - \Gamma_\mu \Gamma_\nu \sin\frac{\theta}{2}$$

$$S^{\mu\nu,-1}(\theta) = \cos\frac{\theta}{2} + \Gamma_{\mu\nu}\Gamma_\nu \sin\frac{\theta}{2}$$

$$S^{\mu\nu}(\theta)\,\Gamma_\mu\, S^{\mu\nu,-1}(\theta) = \left[\cos\frac{\theta}{2} - \Gamma_\mu\Gamma_\nu \sin\frac{\theta}{2}\right]\Gamma_\mu\left[\cos\frac{\theta}{2} + \Gamma_\mu\Gamma_\nu \sin\frac{\theta}{2}\right]$$

$$= \left[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right]\Gamma_\mu + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\Gamma_\nu = \Gamma_\mu\cos\theta + \Gamma_\nu\sin\theta$$

$$S^{\mu\nu}(\theta)\,\Gamma_\nu\, S^{\mu\nu,-1}(\theta) = [\cos\frac{\theta}{2} - \Gamma_\mu\Gamma_\nu\sin\frac{\theta}{2}]\,\Gamma_\nu\,[\cos\frac{\theta}{2} + \Gamma_\mu\Gamma_\nu\sin\frac{\theta}{2}]$$

$$= (\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})\Gamma_\nu - \Gamma_\mu\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = -\Gamma_\mu\sin\theta + \Gamma_\nu\cos\theta$$

or $$S^{\mu\nu}(\theta)\begin{pmatrix}\Gamma_\mu\\ \Gamma_\nu\end{pmatrix}S^{\mu\nu,-1}(\theta) = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\Gamma_\mu\\ \Gamma_\nu\end{bmatrix} = \begin{pmatrix}\Gamma_\mu\cos\theta + \Gamma_\nu\sin\theta\\ -\Gamma_\mu\sin\theta + \Gamma_\nu\cos\theta\end{pmatrix}.$$

Lemma 2: The eigenvalues of $\omega(\mu\nu:\theta)$ are 1 (other axis except $\mu\nu$)

$e^{\pm i\theta}$ (in the $\mu\nu$ plane)

The eigenvalues of $S^{\mu\nu}(\theta)$ are $e^{\pm i\frac{\theta}{2}}$, each 2^{n-1} fold degeneracy. It's becaus $\Gamma_\mu\Gamma_\nu$ eigenvalues are $\pm i$.

lemma 3: Let ω a product of n commuting planar rotations

$\omega = \omega(\alpha\beta|\theta_1)\,\omega(\gamma\delta|\theta_2)\cdots\omega(\mu\nu|\theta_n)$, where $(\alpha\beta\gamma\delta\cdots\mu\nu$ is a permutation of a set of integers. Then

1) $S(\omega) = e^{-\frac{1}{2}\theta_1\Gamma_\alpha\Gamma_\beta}\, e^{-\frac{1}{2}\theta_2\Gamma_\gamma\Gamma_\delta} \cdots e^{-\frac{1}{2}\theta_n\Gamma_\mu\Gamma_\nu}$

② The $2n$ eigenvalues of ω : $e^{\pm i\theta_1}, \quad \cdots \quad e^{\pm i\theta_n}$.

③ The 2^n eigenvalues of $S(\omega)$: $e^{\frac{1}{2}i(\pm\theta_1 \pm \theta_2 \pm \cdots \pm \theta_n)}$.

§ The Solution at $h=0$.

Now let's solve Ising model at $h=0$,

$$\lim_{N\to\infty} \frac{1}{N} \ln Z(0,\beta) = \frac{1}{2} \ln[2\sinh(2\beta J)] + \lim_{n\to\infty} \frac{1}{n} \ln \Lambda$$

where Λ = largest eigenvalue of $V = V_1 V_2$.

$$V_1 = \prod_{k=1}^{n} e^{\theta \tau_k^1}, \text{ with } \theta = \tanh^{-1}[e^{-2\beta J}], \text{ and } V_2 = \prod_{k=1}^{n} e^{\beta J \tau_k^3 \tau_{k+1}^3}.$$

Now we would like to diagonalize V.

In the Representation of Γ-matrix defined before, we have the relation

$$\Gamma_{2m}\Gamma_{2m-1} = 1_1 \otimes \cdots (\tau_m^2 \tau_m^3) \otimes \cdots 1_n = i\, 1_1 \otimes \cdots \otimes \tau_m^1 \otimes \cdots 1_n$$

$$\Rightarrow V_1 = \prod_{k=1}^{n} e^{\theta \tau_k^1} = \prod_{k=1}^{n} e^{-i\theta \Gamma_{2m}\Gamma_{2m-1}}$$

and

$$\begin{aligned} \Gamma_{2m+1}\Gamma_{2m} &= 1_1 \otimes \cdots \otimes \tau_m^1 \tau_m^2 \otimes \tau_{m+1}^3 \otimes \cdots 1_n \\ &= i\, 1_1 \otimes \cdots \otimes \tau_m^3 \otimes \tau_{m+1}^3 \otimes \cdots 1_n \end{aligned}$$

$$\Gamma_1 \Gamma_{2n} = \tau_1^3 \tau_1^1 \otimes \tau_2^1 \cdots \otimes \tau_{n-1}^1 \otimes \tau_n^2$$

$$\begin{aligned} &= -i\{\tau_1^3 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_{n-1} \otimes \tau_n^3\} \\ &\quad \cdot \{\tau_1^1 \otimes \tau_2^1 \otimes \tau_3^1 \cdots \otimes \tau_{n-1}^1 \otimes \tau_n^1\} \end{aligned}$$

$$V_2 = e^{\beta J \tau_n^3 \tau_1^3} \prod_{k=1}^{n-1} e^{\beta J \tau_k^3 \tau_{k+1}^3} = e^{i\beta J \Gamma_{2n+1} \Gamma_1 \Gamma_{2n}} \prod_{k=1}^{n-1} e^{-i\beta J \Gamma_{2k+1}\Gamma_{2k}}$$

$$\Rightarrow \boxed{\begin{array}{l} V = V_2 V_1 = e^{i\phi \Gamma_{2n+1}\Gamma_1\Gamma_{2n}} \prod_{k=1}^{n-1} e^{-i\phi \Gamma_{2k+1}\Gamma_{2k}} \prod_{k=1}^{n} e^{-i\phi \Gamma_{2k}\Gamma_{2k-1}} \\ \phi = \beta J \quad \text{and} \quad \tanh\theta = e^{-2\phi} \end{array}}$$

(chiral decomposition).

Γ_{2n+1} is the chiral matrix:

① $\Gamma_{2n+1}^2 = \mathbb{1}$, $\quad \Gamma_{2n+1}(1+\Gamma_{2n+1}) = 1+\Gamma_{2n+1}$, $\Gamma_{2n+1}(1-\Gamma_{2n+1}) = -(1-\Gamma_{2n+1})$

② $\Gamma_{2n+1} = i^n \Gamma_1 \cdots \Gamma_{2n}$

we use $1 \pm \Gamma_{2n+1}$ to decompose $\boxed{V = \frac{1}{2}(1+\Gamma_{2n+1}) V^+ + \frac{1}{2}(1-\Gamma_{2n+1}) V^-}$

where $V^\pm = e^{\pm i\phi \Gamma_1 \Gamma_{2n}} \prod_{k=1}^{n-1} e^{-i\phi \Gamma_{2k+1}\Gamma_{2k}} \prod_{k=1}^{n} e^{-i\phi \Gamma_{2k}\Gamma_{2k-1}}$

Proof: in V^+ part $\Gamma_{2n+1} = \mathbb{1}$, in V^- part $\Gamma_{2n+1} = -1$. (motivation)

we have $\quad e^{i\phi \Gamma_{2n+1}\Gamma_1\Gamma_{2n}} = \left[\frac{1}{2}(1+\Gamma_{2n+1}) + \frac{1}{2}(1-\Gamma_{2n+1})\right]\left[\cos\phi + i\Gamma_{2n+1}\Gamma_1\Gamma_{2n}\sin\phi\right]$

$$= \frac{1}{2}(1+\Gamma_{2n+1})\left[\cos\phi + i\Gamma_1\Gamma_{2n}\sin\phi\right)$$

$$+ \frac{1}{2}(1-\Gamma_{2n+1})\left[\cos\phi - i\Gamma_1\Gamma_{2n}\sin\phi\right)$$

$$= \frac{1}{2}(1+\Gamma_{2n+1}) e^{i\phi\Gamma_1\Gamma_{2n}} + \frac{1}{2}(1-\Gamma_{2n+1}) e^{-i\phi\Gamma_1\Gamma_{2n}} .$$

Now Γ_{2n+1}, $V^\pm$ commute with each other. We can diagonalize them simultaneously.

Next, we diagonalize $V^{\pm}$ separatly, and for each one of $V^{\pm}$ we obtain 2^n eigen-values. But not all of them belong to the eigenvalue of V. For, V^+ for a particular eigenvalue λ if it's eigenvector belongs to Γ_{2n+1} positive eigenvalue $+1$, then λ is kept, otherwise it's projected out. Similar reasoning applies to V^-.

§ Eigenvalues of V^+ and V^-.

Let's first look at $V^{\pm} = e^{\pm i\phi \Gamma_1 \Gamma_{2n}} \prod_{k=1}^{n-1} e^{-i\phi \Gamma_{2k+1} \Gamma_{2k}} \prod_{k=1}^{n} e^{-i\theta \Gamma_{2k} \Gamma_{2k-1}}$

the corresponding rotation matrix:

$$\Rightarrow \Omega^{\pm} = \omega(1,2n|\mp 2i\phi) \prod_{k=1}^{n-1} \omega(2k+1,2k|2i\phi) \prod_{k=1}^{n} \omega(2k;2k-1|2i\theta)$$

$$= \omega(1,2n|\mp 2i\phi) \prod_{k=1}^{n-1} \omega(2k,2k+1|-2i\phi) \prod_{k=1}^{n} \omega(2k-1;2k|-2i\theta)$$

$\Omega^{\pm}$ are not $V^{\pm}$, but the rotation matrix in a $2n$ dimensional space and then we reduce the diagonalization of $2^n \times 2^n$ matrix to $2n \times 2n$ matrix!

define $\Delta = \prod_{k=1}^{n} \omega(2k-1,2k|-i\theta)$

and $\omega^{\pm} = \Delta \Omega^{\pm} \Delta^{-1} = \Delta \chi^{\pm} \Delta$

where $\chi^{\pm} = \omega(1,2n|\mp 2i\phi) \prod_{k=1}^{n-1} \omega(2k,2k+1|-2i\phi)$

$$= \omega(1,2n|\mp 2i\phi) \left[\omega(23|-2i\phi)\,\omega(45|-2i\phi) \cdots \omega(2n-2,2n-1|-2i\phi)\right]$$

$$\Delta = \omega(12|-i\theta)\,\omega(34|-i\theta) \cdots \omega(2n-1,2n|-i\theta)$$

$$\Delta = \begin{bmatrix} J & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & J & \\ & & J & \\ & & & \ddots \end{bmatrix} \qquad J = \begin{bmatrix} \cos(-i\theta) & \sin(-i\theta) \\ -\sin(-i\theta) & \cos(-i\theta) \end{bmatrix} = \begin{bmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{bmatrix}$$

$$\chi^{\pm} = \begin{bmatrix} a & 0 \quad 0 & & \pm b \\ \begin{matrix} 0 \\ 0 \end{matrix} & K & & \\ & & K & \\ & & & \ddots \\ \mp b & & & a \end{bmatrix} \qquad K = \begin{bmatrix} \cos(-2i\phi) & \sin(-2i\phi) \\ -\sin(-2i\phi) & \cos(-2i\phi) \end{bmatrix} = \begin{bmatrix} \cosh 2\phi & -i\sinh 2\phi \\ i\sinh 2\phi & \cosh 2\phi \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} \cosh 2\phi & -i\sinh 2\phi \\ i\sinh 2\phi & \cosh 2\phi \end{bmatrix}$$

then perform the matrix product

$$\Delta\chi^{\pm}\Delta =$$

$$\begin{bmatrix} \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} & & \\ & \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} & \\ & & \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} \end{bmatrix} \begin{bmatrix} ch2\phi & 0 \quad 0 & & \mp ish2\phi \\ & \begin{matrix} ch2\phi & -ish2\phi \\ ish2\phi & ch2\phi \end{matrix} & & \\ & & \begin{matrix} ch2\phi & -ish2\phi \\ ish2\phi & ch2\phi \end{matrix} & \\ \pm ish2\phi & & & ch2\phi \end{bmatrix} \begin{bmatrix} \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} & & \\ & \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} & \\ & & \begin{matrix} ch\theta & -ish\theta \\ ish\theta & ch\theta \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} A & B & \mp B^{\dagger} \\ B^{\dagger} & A & B \\ \mp B & B^{\dagger} & A \end{bmatrix}$$

with $A = \begin{pmatrix} \cosh 2\phi \cosh 2\theta, & -i\cosh 2\phi \sinh 2\theta \\ i\cosh 2\phi \sinh 2\theta & \cosh 2\phi \cosh 2\theta \end{pmatrix}$

$B = \begin{pmatrix} -\frac{1}{2}\sinh 2\phi \sinh 2\theta & i\sinh 2\phi \sinh^2\theta \\ -i\sinh 2\phi \cosh^2\theta & -\frac{1}{2}\sinh 2\phi \sinh 2\theta \end{pmatrix}$

(checked by mathematica)

More generally, we have $\omega^{\pm} = \Delta \chi^{\pm} \Delta$

$$\begin{bmatrix} A & B & & & & \mp B^{\dagger} \\ B^{\dagger} & A & B & & & \vdots \\ \vdots & B^{\dagger} & A & B & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & A & B \\ \mp B & \cdots & & & B^{\dagger} & A \end{bmatrix}$$

These matrixes have a very nice form of tri-diagonal form, can be mapped to tight binding models with periodical/anti-periodical boundary conditions.

try eigenvector $\psi^{\pm} = \begin{bmatrix} e^{ik} u \\ e^{i2k} u \\ \vdots \\ e^{ink} u \end{bmatrix}$ $\quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

① For ω^{+}, it corresponds to anti-periodical boundary condition $kn = (2m+1)\pi$

or $k = \frac{2m+1}{n}\pi$; $2m+1 = 1, 3, 5, \cdots$

For ω^{-}, it corresponds to periodical boundary condition, $kn = 2m\pi$

or $k = \frac{2m}{n}\pi$ or $2m = 0, 2, 4, \cdots$.

② For each value of e^{ik}, we use Bloch theorem, and arrive at the eigen-equation

$$B^{\dagger} e^{ik} u + A e^{i2k} u + B e^{i3k} u = \lambda_k e^{i2k} u$$

$$\Rightarrow \boxed{\left[A + e^{-ik_m} B^{\dagger} + e^{ik_m} B\right] u = \lambda_{k_m} u}$$ for $k = \frac{\pi}{n} \cdot m$

$m = 0, 2, \cdots 2n-2$

$1, 3, \cdots 2n-1$.

eigen-equation to determine eigenvalues λ_{k_m}

$$A + e^{ik}B + e^{-ik}B^{\dagger} = \quad ,$$

$$= \begin{pmatrix} \cosh 2\phi \cosh 2\theta - \cos k \sinh 2\phi \sinh 2\theta, & -\sin 2\phi \sin k + i[\sinh 2\phi \cosh 2\theta \cos k - \cosh 2\phi \sin 2\theta] \\ -\sin 2\phi \sin k - i(\sinh 2\phi \cosh 2\theta \cos k - \cosh 2\phi \sin 2\theta), & \cosh 2\phi \cosh 2\theta - \cos k \sinh 2\phi \sinh 2\theta \end{pmatrix}$$

$$\det[A + e^{ik}B + e^{-ik}B^{\dagger}] = [\cosh 2\phi \cosh 2\theta - \cos k \sinh 2\phi \sinh 2\theta]^2$$
$$- (\sinh 2\phi \sin k)^2 - \quad .$$
$$- (\cos 2\phi \sinh 2\theta - \cos k \sinh 2\phi \cosh 2\theta]^2$$

$$= \cosh^2 2\phi \, [\cosh^2 2\theta - \sinh^2 2\theta] + \cos^2 k \sinh^2 2\phi \, (\sinh^2 2\theta - \cosh^2 2\theta] - \sinh^2 2\phi \sin^2 k$$

$$= \cosh^2 2\phi - \sinh^2 2\phi = 1$$

Because $A + e^{ik}B + e^{-ik}B^{\dagger}$ are Hermitian, thus their eigenvalues are real.

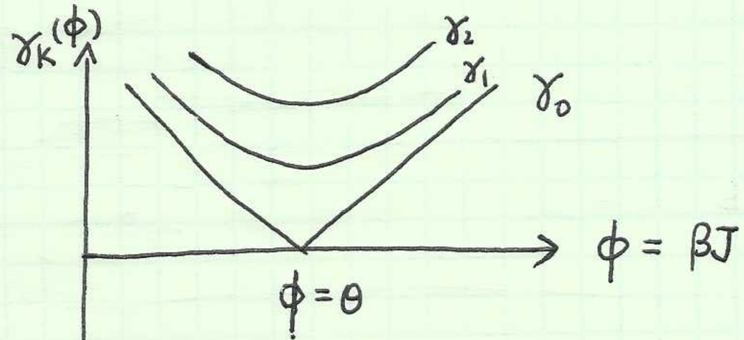
we set $\lambda_{k_m} = e^{\pm \gamma_{k_m}}$ for ($k = \frac{\pi}{n} m$, and $m = 0, \cdots 2n-1$).

$$\Rightarrow tr[A + e^{ik_m}B + e^{-ik_m}B^{\dagger}] = 2\cosh \gamma_{k_m}$$

i.e.
$$\boxed{\begin{aligned} &\cosh \gamma_m = \cosh 2\phi \cosh 2\theta - \cos \frac{\pi m}{n} \sinh 2\phi \sinh 2\theta \\ &m = 0, \cdots, 2n-1 \end{aligned}}$$

Simplify notation
$\gamma_{k_m} \rightarrow \gamma_m$

$\gamma_m = \gamma_{2n-m}$, and also $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_n$



The eigenvalues of V^+ are $e^{\frac{1}{2}(\pm\gamma_0 \pm \gamma_2 \pm \cdots \pm \gamma_{2n-2})}$

V^- are $e^{\frac{1}{2}(\pm\gamma_1 \pm \gamma_3 \pm \cdots \pm \gamma_{2n-1})}$.

As said before, only half eigenvalues belong to eigenvalues of V. As analysed in Kerson Hung $P_{385-386}$, the largest eigenvalue actually is

$$\Lambda = e^{\frac{1}{2}(\gamma_1 + \gamma_3 + \cdots \gamma_{2n-1})} \quad \text{of } V^{-1}.$$

Now

$$\lim_{n\to\infty} \frac{1}{n} \ln \Lambda = \lim_{n\to\infty} \frac{1}{2n}(\gamma_1 + \gamma_3 + \cdots \gamma_{2n-1})$$

set $\nu = \frac{\pi}{n}(2k-1)$

$$\sum_{k=1}^{n} \gamma_{2k-1} \to \frac{n}{2\pi}\int_0^{2\pi} d\nu \; \gamma(\nu) \leftarrow \frac{n}{2\pi}\sum_{k=1}^{n}\frac{2\pi}{n}\gamma_{\nu_k}$$

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\ln\Lambda = \frac{1}{4\pi}\int_0^{2\pi} d\nu\; \gamma(\nu) = \frac{1}{2\pi}\int_0^{\pi} d\nu\; \gamma(\nu)}$$

$$\cosh\gamma(\nu) = \cosh 2\phi \cosh 2\theta - \cos\nu \sinh 2\phi \sinh 2\theta$$

with $\underbrace{\phi = \beta J, \quad \tanh\theta = e^{-2\beta J}}_{\text{Isotropic case}} = e^{-2\phi} \Rightarrow \begin{cases} \sinh 2\theta = \dfrac{1}{\sinh 2\phi} \\ \cosh 2\theta = \coth 2\phi \end{cases}$

$\Rightarrow$

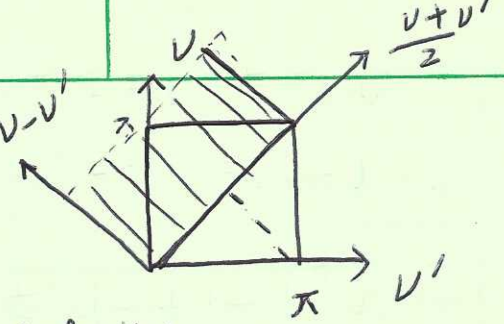
$$\cosh\gamma(\nu) = \cosh 2\phi \coth 2\phi - \cos\nu$$

there's an identity: $\boxed{|z| = \frac{1}{\pi}\int_0^{\pi} dt \; \ln\left(2\cosh z - 2\cos t\right)}$

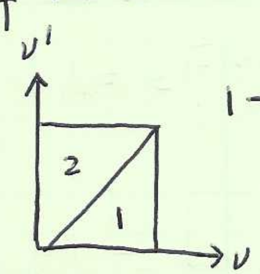
$$\Rightarrow \gamma(\nu) = \frac{1}{\pi}\int_0^{\pi} d\nu' \; \ln\left(2\cosh 2\phi \coth 2\phi - 2\cos\nu - 2\cos\nu'\right)$$

and $\boxed{\lim_{n\to\infty}\frac{1}{n}\ln\Lambda = \frac{1}{2\pi^2}\int_0^{\pi} d\nu \int_0^{\pi} d\nu' \; \ln(2\cosh 2\phi \coth 2\phi - 2\cos\nu - 2\cos\nu')}$

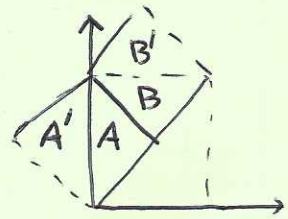
$$\begin{cases} 0 \le \dfrac{\nu+\nu'}{2} \le \pi \\ 0 \le \nu-\nu' \le \pi \end{cases}$$



The integral of the square is equivalent to the that of the shaded rectangle.



$1 \to 2$ by exchanging ν and ν'



$A \to A'$ by $\nu' \to -\nu'$

$B \to B'$ by $\nu \to 2\pi - \nu$

defin $\delta_1 = \dfrac{\nu+\nu'}{2}$, $\delta_2 = \nu - \nu'$

$$\lim_{n\to\infty} \frac{1}{n} \ln \Lambda = \frac{1}{2\pi^2} \int_0^{\pi} d\delta_1 \int_0^{\pi} d\delta_2 \ \ln\left(2\cosh 2\phi \coth 2\phi - 4\cos\delta_1 \cos\tfrac{1}{2}\delta_2\right)$$

$$= \frac{1}{\pi^2} \int_0^{\pi} d\delta_1 \int_0^{\pi/2} d\delta_2 \ \ln\left(2\cosh 2\phi \coth 2\phi - 4\cos\delta_1 \cos\delta_2\right)$$

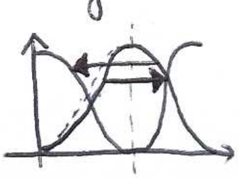
$$= \frac{1}{\pi^2} \int_0^{\pi} d\delta_1 \int_0^{\pi/2} d\delta_2 \ \ln(2\cos\delta_2) + \frac{1}{\pi^2} \int_0^{\pi} d\delta_1 \int_0^{\pi/2} d\delta_2 \ \ln\left(\frac{D}{\cos\delta_2} - 2\cos\delta_1\right)$$

$$= \frac{1}{\pi} \int_0^{\pi/2} d\delta_2 \ln(2\cos\delta_2) + \frac{1}{\pi} \int_0^{\pi/2} d\delta_2 \ \cosh^{-1} \frac{D}{2\cos\delta_2} \qquad \text{where } D = \cosh 2\phi \coth 2\phi$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \Rightarrow \cosh^{-1} \frac{D}{2\cos\delta_2} = \ln\left(\frac{D}{2\cos\delta_2} + \sqrt{\left(\frac{D}{2\cos\delta_2}\right)^2 - 1}\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \ln \Lambda = \frac{1}{2\pi} \int_0^{\pi} d\delta \ \ln\left[D\left(1 + \sqrt{1 - \left(\frac{2}{D}\right)^2 \cos^2\delta}\right)\right]$$

we can change $\cos^2\delta \to \sin^2\delta$ in the intergrand



$$\lim_{n\to\infty} \frac{1}{n}\ln\Lambda = \frac{1}{2}\ln\left(\frac{2\cosh^2 2\beta J}{\sinh 2\beta J}\right) + \frac{1}{2\pi}\int_0^{\pi} d\phi \ \ln\left[\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\phi}\right)\right]$$

$$\kappa = \frac{2}{\cosh 2\phi \coth 2\phi}$$

change symbol from $\delta \to \phi$ for simplicity!

§ Thermodynamic functions

Free energy per site $F = -\frac{1}{N\beta}\ln Z = -\frac{1}{\beta}\left[\ln 2\sinh 2\beta J\right] - \frac{1}{\beta}\lim_{n\to\infty}\frac{\ln\Lambda}{n}$

$$= -\frac{1}{\beta}\ \frac{1}{2}\ln\left[4\cosh^2 2\beta J\right] - \frac{1}{\beta 2\pi}\int_0^{\pi} d\phi \ \ln\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\phi}\right)$$

$$\boxed{F = -\frac{1}{\beta}\ln 2\cosh 2\beta J - \frac{1}{2\pi\beta}\int_0^{\pi} d\phi\ \ln\left[\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\phi}\right)\right], \quad \kappa = \frac{2}{\sinh 2\phi + \frac{1}{\sinh 2\phi}} \le 1}$$

internal energy:

$$u(h=0,\beta) = \frac{d}{d\beta}\left[\beta F\right] = -2J\tanh 2\beta J + \frac{\kappa}{2\pi}\frac{d\kappa}{d\beta}\int_0^{\pi} d\phi\ \frac{\sin^2\phi}{\Delta(1+\Delta)}$$

where $\Delta = \sqrt{1-\kappa^2\sin^2\phi} \Rightarrow \sin^2\phi = \frac{1-\Delta^2}{\kappa^2}$

$$\int_0^{\pi} d\phi\ \frac{\sin^2\phi}{\Delta(1+\Delta)} = \int_0^{\pi} d\phi\ \frac{1}{\kappa^2}\frac{1-\Delta}{\Delta} = -\frac{\pi}{\kappa^2} + \frac{1}{\kappa^2}\int_0^{\pi}\frac{d\phi}{\Delta}$$

$$\Rightarrow u(h=0,\beta) = -2J\tanh 2\beta J + \frac{1}{2\kappa}\frac{d\kappa}{d\beta}\left[-1 + \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}\right]$$

$$\frac{1}{\kappa}\frac{d\kappa}{d\beta} = -\frac{\cosh 2\phi\coth 2\phi}{(\cosh 2\phi \coth 2\phi)^2}\left[\sinh 2\phi\coth 2\phi - \frac{\cosh 2\phi}{\sinh^2 2\phi}\right] 2J$$

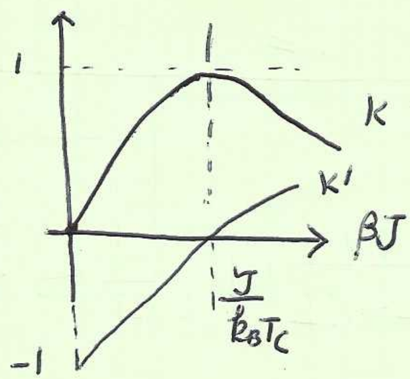
$$= -2J\left[\tanh 2\phi - \frac{1}{sh^2 2\phi \coth 2\phi}\right]$$

$$= -2J\left[\tanh 2\phi - \frac{ch^2 - sh^2}{sh^2}\tan 2\phi\right] = -2J\left[2\tan 2\phi - \coth^2 2\phi \cdot \tan 2\phi\right]$$

$$= -2J\left[2\tan 2\phi - \coth 2\phi\right]$$

$$\Rightarrow -2J\tan 2\phi - \frac{1}{2k}\frac{dk}{d\beta} = -J\coth 2\phi$$

$$\Rightarrow \boxed{\begin{array}{l} \mathcal{U}(0,\beta) = -J\coth 2\beta J\left[1 + \frac{2}{\pi}k' K_1(k)\right] \\ K_1(k) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \quad \text{and} \quad k' = 2\tanh^2 2\beta J - 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad k = \frac{2\sinh 2\beta J}{\cosh^2 2\beta J} \\ \text{with } k^2 + k'^2 = 1 \end{array}}$$



Then the specific heat

$$\frac{1}{k_B}C(0,T) = -\frac{1}{\beta^2}\frac{d}{d\beta}\mathcal{U}(0,\beta)]$$

$$= \frac{J}{\beta^2}\frac{d}{d\beta}\left[\coth 2\beta J\left[1 + \frac{2}{\pi}k' K_1(k)\right]\right]$$

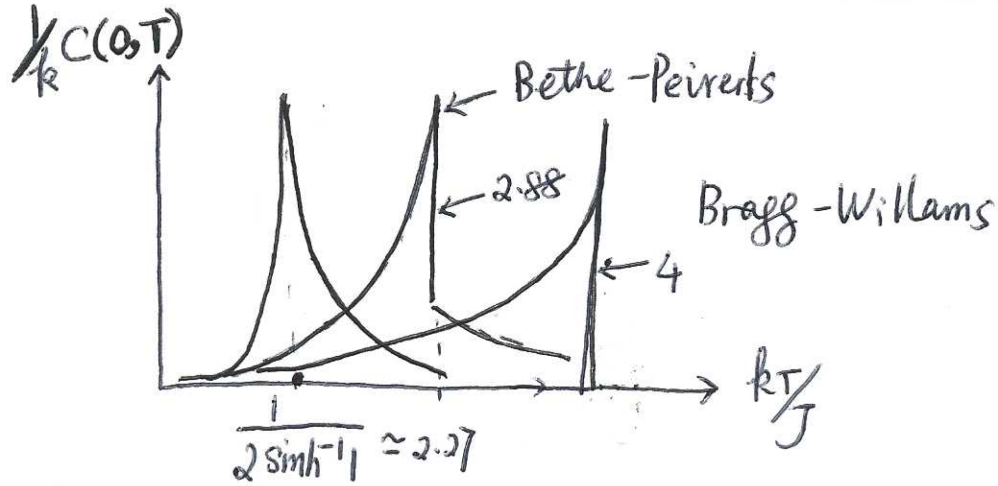
should be checked later, too tired now.

$$\longrightarrow \boxed{\frac{2}{\pi}(\beta J\coth 2\beta J)^2\left[2K_1(k) - 2E_1(k) - (1-k')\left[\frac{\pi}{2} + k' K_1(k)\right]\right]}$$

The singularity comes from $K_1(k)$ at $k=1$, i.e

$$K_1(k) \simeq \ln\frac{4}{\sqrt{1-k^2}} \quad \text{and} \quad k_c = 1 \Rightarrow 2\sinh 2\beta J = 1 + \sinh^2 2\beta J$$

$$\begin{cases} \sinh^2 2\beta_c J = 1 \\ \cosh^2 2\beta_c J = 2 \end{cases} \Rightarrow e^{-J\beta_c} = \sqrt{2} - 1$$



$$\frac{1}{k_B}C(0,T) \approx \frac{2}{\pi}\left(\frac{2J}{k_B T_c}\right)^2\left[-\ln\left|1-\frac{T}{T_c}\right|\right] + \cdots$$

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$$m(0,T) = \left\{\left[1-(\sinh 2\beta J)^{-4}\right]\right\}^{1/8} \qquad T < T_c$$