Continous field theory — Ginsburg-Landau free energy

In the Ising model, the variables are defined in the lattice, which is not convenient. As approaching phase transition, long wave length fluctuations become more and more important such that we can neglect the discrete nature of the lattice, but build up a continous theory. The only things importante are dimensionality and the symmetry class, which determine the universal class.

For Ising model, with an magnetization density $\frac{M}{N}=m$, let's calculate it's free enenergy density,

$$\frac{1}{N} E(m) \approx -\frac{z}{2} J m^2, \quad \text{(coarse averaging)}$$

$\ln n! \simeq n\ln n - n$

The degenergy numbe $\dfrac{N!}{\left[\frac{N}{2}(1+m)!\right]\left[\frac{N}{2}(1-m)!\right]} = g$

$$F(m,T) = E(m) - \frac{k_B T \ln g}{N} = -\frac{z}{2} J m^2 + k_B T\left[\left(\ln\frac{1+m}{2}\right)\frac{1+m}{2} + \left(\ln\frac{1-m}{2}\right)\frac{1-m}{2}\right]$$

Expanding around $m \approx 0$, we have

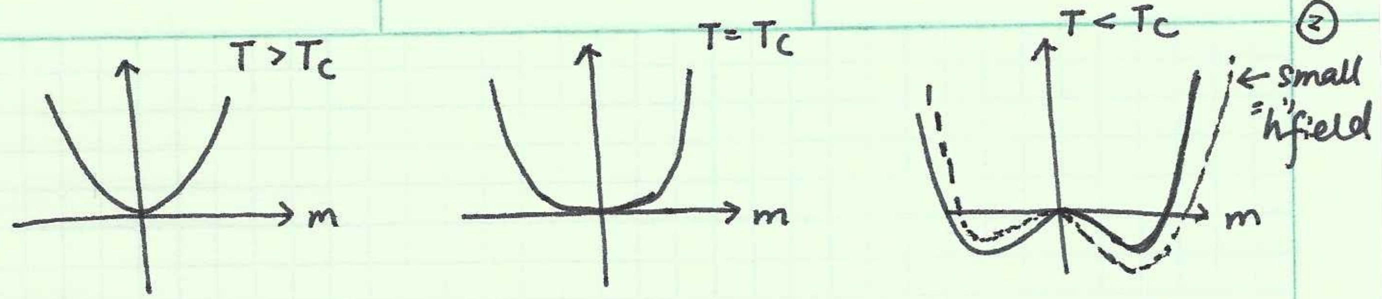
$$F(m,T) = \left(-\frac{z}{2}J + \frac{k_B T}{2}\right) m^2 + \frac{k_B T}{12} m^4 + \cdots$$

$$- k_B T \ln 2$$

$$= \frac{k_B T_c}{2}\left[\frac{T}{T_c} - 1\right] m^2 + \frac{k_B T}{12} m^4 + \cdots$$

$$\beta F(m,T) = \frac{1}{2}\left(1 - \frac{T_c}{T}\right) m^2 + \frac{m^4}{12} + \cdots .$$

(m is call order parameter)



This model is often called φ^4-theory. The symmetry property is the same as Ising model. — Z_2 symmetry, i.e. $m \to -m$. At $T < T_c$, spontaneous symmetry breaking occurs, and $\langle m \rangle \neq 0$. More rigorously.

$$\lim_{h\to 0} \lim_{V\to\infty} m(h, V) \neq 0 .$$

The sequence of limits cannot be exchanged! otherwise

$$\lim_{h\to 0} m(h, V) = 0$$ at any finite values of V!

As $V \to \infty$, the tunneling from one ground state (vaccum) to another one need to pass the barrier energy proportional to V, thus pratically the tunneling probability $\to 0$, which leads to spontaneous symmetry breaking!

If the order parameter is spatially varying, it costs energy, and we add an extra term as

$$\boxed{\beta F = \gamma (\nabla m)^2 + \frac{1}{2}\alpha(T)\, m^2 + \frac{1}{4}\beta\, m^4}$$

This is the standard φ^4 theory for the Ising class. If for a Heisenberg spin. we need to set m as a vector spin density $\vec{m}$,

$$\beta F = \gamma \left(\nabla . \vec{m}\right)^2 + \frac{1}{2}\alpha(T)\, \vec{m}\cdot\vec{m} + \frac{1}{4}\beta \left(\vec{m}\cdot\vec{m}\right)^2, \quad \leftarrow \text{SO(3) model}$$

where $\quad \alpha(T) = \alpha_0 \left(1 - \frac{T_c}{T}\right).$

Critical exponents (mean field level)

Define a dimensionless ~~temperature~~ $t = \frac{T}{T_c} - 1 \simeq 1 - \frac{T_c}{T}$ around $T \sim T_c$.

1. $\boxed{\alpha: \text{ specific heat} \quad C(t) \sim C_\pm |t|^{-\alpha_\pm}}$ as $t \to 0^\pm$.

$$C = -T \frac{\partial^2 F}{\partial T^2}. \qquad \text{denote } V = \beta F, \Rightarrow \frac{\partial^2 F}{\partial T^2} = 2k_B \frac{\partial V_{min}}{\partial T} + k_B \frac{\partial^2 V_{min}}{\partial T^2}$$

$$V = \frac{1}{2}\alpha_0 \left(\frac{T}{T_c} - 1\right) m^2 + \frac{1}{4}\beta m^4$$

the saddle point solution: $m_0 = \begin{cases} 0 & t > 0 \\ \sqrt{\frac{\alpha_0 (1 - T/T_c)}{\beta}} & t < 0 \end{cases}$.

$$\Rightarrow V_{min} = \begin{cases} 0, & t > 0, \\ -\frac{3}{8}\frac{\alpha_0^2 \left(\frac{T}{T_c} - 1\right)^2}{\beta} & t < 0. \end{cases} \Rightarrow C = \begin{cases} 0, & t > 0 \\ \frac{3k_B}{4}\frac{\alpha_0^2}{\beta}\frac{T}{T_c}\left[\frac{2T}{T_c} - 1\right], & t < 0. \end{cases}$$

$\Rightarrow C$ is finite but discontinous, thus $\alpha_\pm = 0$ at mean field level.

2. $\boxed{\beta: \text{ spontaneous magnetization.} \quad M \sim (-t)^\beta, \quad t < 0.}$

mean field results $\beta = 1/2$.

3 $\boxed{\gamma: \text{ magnetic susceptibility}: \quad \chi^\pm \sim |t|^{-\gamma_\pm}}$

$$V(h) = \frac{1}{2}\alpha m^2 + \frac{1}{4}\beta m^4 - hm. \Rightarrow \frac{\partial V}{\partial m} = \alpha m + \beta m^3 - h = 0$$

at $\alpha > 0$, $m = h/\alpha$ + higher order $\Rightarrow \chi \sim t^{-1} \Rightarrow \gamma_+ = 1$.

$$\alpha < 0 \qquad \alpha + \beta m^2 = \frac{h}{m} \Rightarrow m^2 = \left(-\frac{\alpha}{\beta}\right)\left(1 + \frac{h}{|\alpha| m}\right)$$

$$m = \left[\frac{|\alpha|}{\beta}\right]^{1/2}\left(1 + \frac{h}{2|\alpha|\left(\frac{|\alpha|}{\beta}\right)^{1/2}}\right] \simeq \left(\frac{|\alpha|}{\beta}\right)^{1/2} + \frac{h}{2|\alpha|} \Rightarrow \chi_- \sim |t|^{-1}$$

and $\gamma_- = 1$.

For the SO(3) model, in the symmetry broken phase. The susceptibility is anisotropic. The above calculation works for the longitudinal susceptibility, i.e. $\vec{h} \parallel \vec{M}$. The transverse magnetization $\Delta\vec{M} \perp \vec{M}$ does not cost energy at $(\Delta M)^2$ level.

$$\vec{M} = (m_0 + \delta m_{\parallel})\hat{x} + \delta\vec{m}_{\perp}$$

$$\Rightarrow M^2 = m_0^2 + 2m_0\delta m_{\parallel} + (\delta m_{\parallel})^2 + (\delta m_{\perp})^2$$

$$M^4 = m_0^4 + 4m_0^2(\delta m_{\parallel})^2 + (\delta m_{\parallel})^4 + (\delta m_{\perp})^4$$
$$+ 4m_0^3\delta m_{\parallel} + 2m_0^2(\delta m_{\parallel})^2 + 2m_0^2(\delta m_{\perp})^2$$
$$+ 4m_0(\delta m_{\parallel})^3 + 4m_0\delta m_{\parallel}(\delta m_{\perp})^2 + 2(\delta m_{\parallel})^2(\delta m_{\perp})^2$$

$$\frac{\alpha M^2}{2} + \frac{\beta}{4}M^4 \simeq (\delta m_{\parallel})^2\left[\frac{\alpha}{2} + \frac{6\beta}{4}m_0^2\right] + (\delta m_{\perp})^2\left[\frac{\alpha}{2} + \frac{\beta}{4}2m_0^2\right] + \cdots$$
$$= (\delta m_{\parallel})^2[|\alpha|] + 0\cdot(\delta m_{\perp})^2 \Rightarrow \boxed{\chi_{\perp}^{-1} = 0} \leftarrow \text{in the entire sym breaking phase.}$$

$\Rightarrow$ the inverse of transverse susceptibility is exactly zero, which is a result of Goldstine theory. Transverse modes are zero modes.

4. $\boxed{\begin{array}{l}\delta\text{: at } T = T_c\text{, the magnetization v.s. external field.}\\ \quad M \sim h^{1/\delta}\end{array}}$

at the mean field level, $\delta = 3$.

5: $\boxed{\begin{array}{l}\nu\text{: Correlation length} \sim \text{temperature.}\\ \quad \xi \sim |T - T_c|^{-\nu_{\pm}}\ .\end{array}}$

MF level $\beta F = \frac{\gamma}{2}(\nabla m)^2 + \frac{\alpha}{2}|m|^2 + \frac{\beta}{4}|m|^4$

① at $\alpha > 0$. $\beta F = \sum_k (\gamma k^2 + \alpha)\frac{1}{2} m(k) m(-k)$

$$\langle m(k) m(-k)\rangle = \frac{\int d\,\mathrm{Re}\,m(k)\, d\,\mathrm{Im}\,m(k)\; m(k)m(-k)\, e^{\cdots}}{\int d\,\mathrm{Re}\,m(k)\, d\,\mathrm{Im}\,m(k)\; e^{-(\gamma k^2+\alpha)\frac{1}{2}m(k)m(-k)}} = \frac{1}{\gamma k^2+\alpha}$$

$$\langle m(r) m(0)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{r}}}{\gamma k^2+\alpha} \sim \frac{1}{r} e^{-r/\xi}, \text{ where } \xi = \sqrt{\frac{\gamma}{\alpha}} \sim \frac{1}{t^{1/2}}.$$

② at $\alpha < 0$, we have similar result. for longitudinal correlation

but $\xi_{\parallel} \sim \sqrt{\frac{\gamma}{2|\alpha|}} \propto \frac{1}{|T-T_c|^{1/2}}$.

The transverse correlation length diverges in the ordered phase.

$$\langle m_\perp(r)\, m_\perp(0)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{r}}}{\gamma k^2} \sim \frac{1}{r}.$$

6: η: two point correlation function at $T = T_c$. $\langle m(r) m(0)\rangle \sim r^{-(d-2+\eta)}$, $\langle m(k) m(-k)\rangle \sim \frac{1}{k^{2-\eta}}$

mean-field results $\eta = 0$.

	2d Ising model	mean field	spherical mode ($n\to\infty$) $4>d>2$
α	$\alpha = 0$ (logrithmic)	$\alpha = 0$ (discont)	$\alpha = \frac{d-4}{d-2}$
β	$\beta = 1/8$	$\beta = 1/2$	$\beta = 1/2$
γ	$\gamma = 7/4$	$\gamma = 1$	$\gamma = \frac{2}{d-2}$
δ	$\delta = 15$	$\delta = 3$	$\delta = \frac{d+2}{d-2}$
ν	$\nu = 1$	$\nu = 1/2$	$\nu = \frac{1}{d-2}$
η	$\eta = 1/4$	$\eta = 0$	$\eta = 0$