★ transfer matrix for 2D Ising model — Kink operator

1 we have already mapped the 1D Ising model

$$Z = \sum_{\{\sigma\}} e^{\beta J \sum_i (\sigma_i\sigma_{i+1}-1)} = \sum_{\sigma_1\cdots\sigma_N} \underbrace{T_{\sigma_1\sigma_2}\cdots T_{\sigma_N\sigma_1}}_{\text{time-evolution}} = \mathrm{tr}\, T^N$$

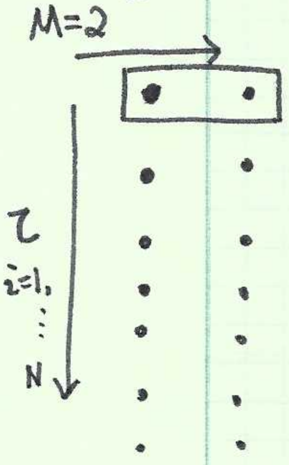
with T is 2×2 matrix defined as $e^{h\Delta\tau\sigma_1}\cdot$ const

$$= \begin{pmatrix}1 & e^{-2\beta J}\\ e^{-2\beta J} & 1\end{pmatrix} \Rightarrow \boxed{Z = \mathrm{tr}\left[e^{N\Delta\tau h\sigma_1}\right] = \mathrm{tr}\left[e^{-\beta_\tau h\sigma_1}\right]}$$

with $\beta_\tau = N\Delta\tau \xrightarrow{N\to\infty} \infty$

Relation $\boxed{\sinh 2\beta J \, \sinh h\Delta\tau = 1}$

2 let us consider the case of two-chain



$$Z = \sum_{\{S\}} e^{\beta J \sum_{i=1}^{N} \sum_{j=1}^{2} S(i,j)S(i,j+1) + S(i,j)S(i+1,j)}$$

I change classic variable notation to $S(i,j)$

The transfer matrix is 4-dimensional

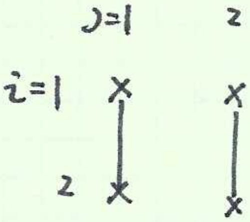
define $S_i = (S(i,1), S(i,2))$, which takes 4-possible values $S_i = (1\ 1), (-1\ 1)\ (1,-1)$, and $(-1,-1)$, Then

$$Z = \sum_{S_1,\cdots S_N} T_{S_1S_2}\, T_{S_2S_3} \cdots T_{S_NS_1} = \mathrm{tr}\, T^N$$

$$T_{S_1 S_2} = e^{\beta J \sum_{j=1}^{2} S(1j)\, S(2j)} \cdot e^{\beta J\, S(1,1)\, S(1,2)}$$

$$= \left[T'\right]_{S_1 S_2} \left[T''\right]_{S_2 S_2}$$

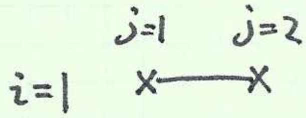
T' describes the two vertical bonds: independent evolution of two spins



$$T' = \begin{array}{c|c} S_1 \backslash S_2 & (1\ 1)\ \ (-1\ 1)\ \ (1\ -1)\ \ (-1\ -1) \\ \hline \begin{array}{c}(1\ 1)\\(-1\ 1)\\(1\ -1)\\(-1\ -1)\end{array} & \left[\begin{array}{cccc} & & & \\ & \cdots & & \\ & & & \\ & & & \end{array}\right] \end{array} = \left[e^{h\Delta\tau\, \sigma_1(j=1)} \otimes e^{h\Delta\tau\, \sigma_1(j=2)}\right]_{S_1 S_2}$$

$$= e^{h\Delta\tau \sum_{j=1}^{2} \sigma_1(j)}$$

T'': $\left[\quad\right]$ is diagonal: vertical bond



$$\Rightarrow T'' = \left[e^{\beta J\, \sigma_3(1)\, \sigma_3(2)}\right]_{S_2 S_2}$$

$$T_{S_1 S_2} = \left[e^{h\Delta\tau \sum_{j=1}^{2} \sigma_1(j) + \beta J \, \sigma_3(1)\,\sigma_3(2)} \right]_{S_1 S_2}$$

This picture can be generalized to M-chains, and Tmatrix represent , time-evolution of M spins, and thus T becomes $2^M \times 2^M$ dimensional. If we use periodical boundary condition along the j-direction. we have

$$T = e^{h\Delta\tau \sum_{j=1}^{M} \sigma_1(j) + \beta J \sum_{j=1}^{M} \sigma_3(j)\sigma_3(j+1)}$$

$$= e^{-\Delta\tau H}$$

and

$$\boxed{\begin{array}{ll} H = -h \sum_{j=1}^{M} \sigma_1(j) - K \sum_{j=1}^{M} \sigma_3(j)\sigma_3(j+1) & \\ K/h = \frac{\beta J}{h\Delta\tau} & \text{1D transferse} \\ \sinh 2\beta J \, \sinh h\Delta\tau = 1 & \text{field Ising model.} \end{array}}$$

$$\Rightarrow Z = \mathrm{tr}\left[e^{-\tau H}\right]$$

where $\tau = N\Delta\tau \to \infty$.

$$\boxed{\text{we have mapped a 2D classic problem to 1D QM problem!}}$$

2D classical phase transition $\longrightarrow$ 1D Quantum phase transition.

Now let's treat $H = -K \sum_i \left(g\,\sigma_1(i) + \sigma_z(i)\,\sigma_z(i+1) \right)$

as a quantum model, and consider it's ground state properties.

1 strong coupling limit $g \gg 1$

If $g \to \infty$, the ground state is a paramagnetic state with each site spin parallel to $\hat{x}$-direction.

$$|\Omega\rangle = \prod_i |\rightarrow\rangle_i \, , \quad \text{and} \quad \langle \Omega | \sigma_i^z \, \sigma_j^z | \Omega \rangle = \delta_{ij} .$$

If g is large but finite, we expect $\langle \Omega | \sigma_i^z \sigma_j^z | \Omega \rangle \sim e^{-|x_i - x_j|/\xi}$,

i.e. short-range correlated. The excitation is to flip one site spin to $\leftarrow$, i.e.

$$\underset{}{\rightarrow} \; \rightarrow \cdots \underset{i}{\leftarrow} \; \rightarrow \; \rightarrow \qquad \rightarrow \quad \boxed{|i\rangle = |\leftarrow\rangle_i \prod_{j \neq i} |\rightarrow\rangle_j .}$$

All the states $|i\rangle$ are degenerate at the limit $g \to +\infty$. At $1/g$ level, the $\sigma_z \cdot \sigma_z$ term couples different states together as

$$\boxed{\langle i | -K \sum_n \sigma_z(n)\,\sigma_z(n+1) \, | i \pm 1 \rangle = -K}$$

we can form $|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} \, |j\rangle$. its eigen energy is

$$\boxed{\varepsilon_k = Kg \left[2 - \tfrac{2}{g} \cos(k) + O(1/g^2) + \cdots \right]}$$

③ weak coupling $g \ll 1$

two fold degeneracy $|\uparrow\rangle \otimes |\uparrow\rangle, \cdots$ and $|\downarrow\rangle \otimes |\downarrow\rangle \cdots \otimes |\downarrow\rangle$.

σ^z has long-range order. The low energy excited states are topological nature — kink.

$$|\uparrow\rangle \otimes |\uparrow\rangle \cdots \underset{i}{|\uparrow\rangle} \otimes \underset{i+1}{|\downarrow\rangle} \otimes |\downarrow\rangle \cdots$$

if we neglect the coupling between sectors with different number of kinks. we can easily work out its energy dispersion

$$\boxed{\mathcal{E}_k = K(2 - 2g\cos ka + O(g^2))}$$

[the $Kg\sigma_1$ term builds up hopping of kinks].

③ Exact solution of spectrum

Define non-local transformation: Jordan–Wigner transformation

$$\sigma_i^z = 1 - 2C_i^\dagger C_i$$

$$\sigma_i^+ = \prod_{j<i} (1 - 2C_j^\dagger C_j)\, C_i$$

$$\sigma_i^- = \prod_{j<i} (1 - 2C_j^\dagger C_j)\, C_i^\dagger$$

$\xrightarrow{\text{inverse}}$

$$C_i = \Big(\prod_{j<i} \sigma_j^z\Big)\sigma_i^+$$

$$C_i^\dagger = \Big(\prod_{j<i} \sigma_j^z\Big)\sigma_i^-$$

> Ex: please check that $\{C_i, C_j^\dagger\} = \delta_{ij}$, and thus $C_i, C_i^\dagger$ are spinless fermion operators.

For transverse field Ising model, it's more convenient to do a further transform $\sigma^z \to \sigma^x$ and $\sigma^x \to -\sigma^z$.

Such that

$$\sigma_i^x = 1 - 2c_i^\dagger c_i$$

$$\sigma_i^z = -\prod_{j<i}(1-2c_j^\dagger c_j)(c_i + c_i^\dagger)$$

$$\Rightarrow \quad H = -K\sum_i\{g(1-2c_i^\dagger c_i) + (c_i + c_i^\dagger)(c_{i+1} + c_{i+1}^\dagger)\}$$

$$= -K\sum_i\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i - 2g\,c_i^\dagger c_i - g\right)$$

$$= K\sum_k\left(2(g-\cos k)\,c_k^\dagger c_k - 2i\sin k\,(c_{-k}^\dagger c_k^\dagger - c_k c_{-k}) - g\right]$$

$$= K\sum_k \begin{pmatrix} c_k^\dagger & c_{-k}\end{pmatrix}\begin{bmatrix} 2(g-\cos k) & 2i\sin k \\ -2i\sin k & -2(g-\cos k)\end{bmatrix}\begin{bmatrix} c_k \\ c_{-k}^\dagger\end{bmatrix}$$

→ The excitation spectrum

$$\varepsilon_k = 2K\left((g-\cos k)^2 + \sin^2 k\right)^{1/2} = 2K\left(1+g^2-2g\cos k\right)^{1/2}$$

Ex: ① please diagonalize the above matrix by Bogoliubov transformation

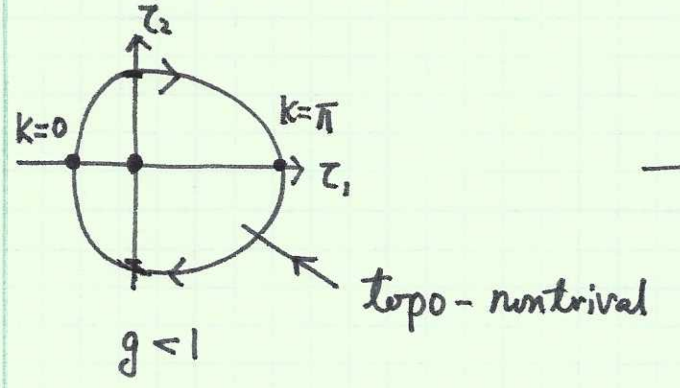
② check that ε_k at $g \ll 1$ and $g \gg 1$, agrees with the approximate expression given above.

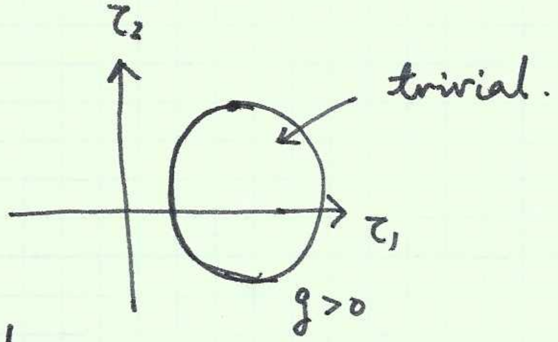
At both $g>1$, and $g<1$, because $1+g^2 > 2g$, the spectra of ε_k is gapped. But at $g=1$, $\varepsilon_k = 4K\left|\sin\frac{k}{2}\right|$, the spectra is gapless, which indicate a quantum phase transition. Indeed, $|g|<1$ corresponds to topological pairing, and $|g|>1$ is topologically trivial pairing!

The pairing matrix $\Delta_k = 2[(g-\cos k)\,\tau_1 - \sin k\,\tau_2]$

as k in the BZ, $k \in [-\pi, \pi]$, if we represent Δ_k as a 2-vector

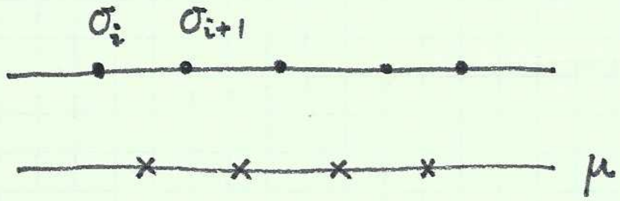
in the basis of τ_1, τ_2, we have





§ Come back to the spin language, we have an order/disorder transition.

Duality (site-bond)



$$\begin{cases} \mu^z_{n+1/2} = \prod_{j=1}^{n} \sigma^x_j \\ \mu^x_{n+1/2} = \sigma^z_n \sigma^z_{n+1} \end{cases} \longrightarrow \begin{cases} \sigma^z_n = \prod_{j=0}^{n-1} \mu^x_{j+1/2} \\ \sigma^x_n = \mu^z_{n-1/2}\, \mu^z_{n+1/2} \end{cases}$$

in terms of $\mu \Rightarrow$

$$H = -K\,[\,g \sum_n \mu^z_{n-1/2}\, \mu^z_{n+1/2} + \mu^x_{n+1/2}\,]$$

$g \to 1/g$. self-duality.

what is μ ? the Kink operator / disorder operator

$$|\Omega\rangle = \prod_n |\uparrow\rangle_n \Rightarrow \mu^z_{n+1/2}|vac\rangle = |\underset{n}{\downarrow\downarrow\cdots\downarrow}\underset{n+1/2}{\uparrow}\uparrow\uparrow\cdots\rangle$$

Thus $g>1$, σ_z disordered, $\longleftrightarrow$ μ^z ordered

<1 σ_z ordered $\longleftrightarrow$ μ^z disordered

self-dual

Further Come back to 2D Ising model $\Longrightarrow$ low $T<T_c$ / $T>T_c$ Winger-Kramer duality.

§ Majarana Representation

$$\xi_1(n) = \frac{C_n^\dagger + C_n}{\sqrt{2}}, \quad \xi_2(n) = \frac{C_n^\dagger - C_n}{-\sqrt{2}\,i} \Rightarrow \{\xi_i, \xi_j\} = \delta_{ij}$$

Ex: please verify that in the Majurana Rep

$$H = K(\quad ig\,\xi_2(n)\,\xi_1(n) - i\,\xi_2(n)\,\xi_2(n+1)]$$

$\rightarrow$ continuum version

$$\frac{H}{K} = -i\,\xi_2(n)\left(\xi_1(n+1) - \xi_1(n)\right) + i(g-1)\,\xi_2(n)\,\xi_1(n)$$

$$\rightarrow \int dx\ \xi_2(-i\partial_x)\xi_1 - im\,\xi_1\xi_2 \qquad m = g-1$$

$$= \frac{1}{2}\int dx\ \xi^T(\alpha p + \beta m)\xi\ , \quad \text{where } \xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},\ \alpha = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$$

$$\beta = \begin{pmatrix} & -i\\ i & \end{pmatrix} \quad p = -i\partial_x$$