Assignment #3

PHSY 217 Spring 2014

Problem 1. (Goldenfeld Exercise 9-1)

Solution:

(a) Notice that the six critical exponents are not independent since there are four scaling laws. The origin of the scaling laws is that all of the critical exponents can be expressed in terms of y_t and y_h. Hence our strategy would be to write δ and η in terms of y_t and y_h, and then derive the relation.

δ is defined to be $m \sim H^{1/\delta}$ $(T = T_c)$. Notice that $m = \langle \hat{m} \rangle = \frac{tr(\hat{m} e^{-\beta \hat{H}})}{tr(e^{-\beta \hat{H}})}$, so that any constant term in the action canceled in numerator and denominator. Thus the scaling of the corresponding singular part of free energy is just

$$f_s(t,h) = \ell^{-d} f_s(t_\ell, h_\ell) = \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h}),$$

which is different from what is investigated in exercise 12-3 (d) where there is additional term that contribute to f_s due to the fact that in total free energy $f = -k_B T \ln Z$ and there will be no such cancelling.

Taking derivative with respect to h,

$$m(t,h) = \ell^{-d+y_h} m(t\ell^{y_t}, h\ell^{y_h})$$

Choose ℓ such that the system is driven to $h\ell^{y_h} = 1$. Then $\ell = h^{-\frac{1}{y_h}}$

$$m(t,h) = h^{\frac{d-y_h}{y_h}} m(th^{-\frac{y_t}{y_h}}, 1)$$

Set $t=0$,

$$m(0,h) = h^{\frac{d-y_h}{y_h}} m(0,1).$$

Hence

$$\frac{1}{\delta} = \frac{d-y_h}{y_h},$$

i.e.

$$\delta = \frac{y_h}{d-y_h}.$$

Then let's derive the expressions for ν and η. Let's work in momentum space and derive the scaling form of G.

By definition,

$$G(\vec{r}, \vec{r}') = G(\vec{r}-\vec{r}') := \langle \phi(\vec{r})\, \phi(\vec{r}') \rangle$$
$$= \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{d^d\vec{k}'}{(2\pi)^d} \langle \phi_{\vec{k}}\, \phi_{-\vec{k}'} \rangle e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}'\cdot\vec{r}'}$$
$$= \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{d^d\vec{k}'}{(2\pi)^d} \hat{G}(\vec{k}) \cdot (2\pi)^d \delta^d(\vec{k}-\vec{k}') e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}'\cdot\vec{r}'}$$
$$= \int \frac{d^d\vec{k}}{(2\pi)^d} \hat{G}(\vec{k}) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Hence $\quad G(\vec{r},\vec{r}') = \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{\langle \phi_{\vec{k}}\, \phi_{-\vec{k}} \rangle}{(2\pi)^d \delta^d(0)} \quad e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$

$$= \frac{1}{Vol} \int \frac{d^d\vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}}\, \phi_{-\vec{k}} \rangle\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Then under RG,

$$G(\vec{r},t,h) = \frac{1}{V} \int \frac{d^d\vec{k}}{(2\pi)^d} \langle \phi'_{\vec{k}'}\, \phi'_{-\vec{k}'} \rangle\, \ell^{-2[\phi]}\, e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{V'}{V}\frac{1}{V'} \int \ell^{-d} \frac{d^d\vec{k}'}{(2\pi)^d} \langle \phi'_{\vec{k}'}\, \phi'_{-\vec{k}'} \rangle\, \ell^{-2[\phi]}\, e^{i\vec{k}'\cdot\vec{r}/\ell}$$

$$= \ell^{-2[\phi]-2d}\, \frac{1}{V'} \int \frac{d^d\vec{k}'}{(2\pi)^d} \langle \phi'_{\vec{k}'}\, \phi'_{-\vec{k}'} \rangle\, e^{i\vec{k}'\cdot\vec{r}/\ell}$$

$$= \ell^{-2[\phi]-2d}\, G(\vec{r}/\ell,\ t_\ell,\ h_\ell)$$

where we've employed scaling of ϕ, k and Volume as

$$\cancel{\phi(\vec{k})} \begin{cases} \vec{k}' = \ell\, \vec{k} \\ \phi'(\vec{k}') = \ell^{[\phi]}\phi(\vec{k}) \\ V' = \ell^{-d}\, V \end{cases}$$

But the scaling of ϕ is related to that of H as $y_h = -[\phi]$, as can be easily seen from $H\int d^d\vec{r}\,\phi(\vec{r}) = H\,\phi(\vec{k}=0)$ in the Hamiltonian.

So $\quad G(\vec{r},t,h) = \ell^{2(y_h-d)}\, G(\vec{r}/\ell,\ t_\ell,\ h_\ell) = \ell^{2(y_h-d)}\, G(\vec{r}/\ell,\ t\,\ell^{y_t},\ h\,\ell^{y_h})$

Actually the above equality is only in the sense of singular part of G. This is because when we do $G(\vec{r}/\ell, t_\ell, h_\ell)$ with respect to $G(\vec{r},t,h)$, modes lying in $[\Lambda/\ell, \Lambda]$ are lost. But they don't contribute to any singular behaviour anyway as long as $\Lambda/\ell \gg 1/\xi$.

Choose ℓ such that $t\,\ell^{y_t} = 1$, we have

$$G(\vec{r},t,h) = t^{2(d-y_h)/y_t}\, G(\vec{r}\,t^{1/y_t},\ 1,\ h\,t^{-y_h/y_t})$$

Write

$$G(\vec{r}\,t^{1/y_t},\ 1,\ h\,\ell^{-y_h/y_t}) = (r\,t^{1/y_t})^{-2(d-y_h)}\, F_G(r\,t^{1/y_t},\ h\,t^{-y_h/y_t})$$

Then

$$G(\vec{r},t,h) = \frac{1}{r^{2(d-y_h)}}\, F_G(r\,t^{1/y_t},\ h\,t^{-y_h/y_t})$$

which recovers scaling hypothesis.

Thus $\quad 2(d-y_h) = d-2+\eta,\quad \nu = \frac{1}{y_t}.$

Combining $\begin{cases} \delta = \dfrac{y_h}{d-y_h} \\ 2(d-y_h) = d-2+\eta \end{cases}$ together, it is straightforward to get

$\delta = \dfrac{d+2-\eta}{d-2+\eta}$ by cancelling y_h.

(b) Again using the same trick to make $|t_\ell|=1$. We have

$$f_s(t,h) = \ell^{-d} f_s(t_\ell, h_\ell)$$
$$= \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h}) \quad (|t|\ell^{y_t}=1 \Rightarrow \ell = |t|^{-\frac{1}{y_t}})$$
$$= |t|^{d/y_t} f_s(\pm 1, h|t|^{-y_h/y_t})$$

Plugging in $\bar{\nu} = 1/y_t$, we have

$$f_s(t,h) = |t|^{d\bar{\nu}} F_f^{\pm}(h|t|^{-\bar{\nu}y_h})$$
$$= |t|^{d\bar{\nu}} F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)$$

where

$$F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right) = f_s(\pm 1, h|t|^{-y_h/y_t})$$

Let's write

$$F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right) = \left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)^{\lambda} \phi_{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)$$

and we choose the power λ such that

$$|t|^{d\bar{\nu}}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)^{\lambda} = h^{\lambda}$$

which means that

$$d\bar{\nu} - \lambda\bar{\nu}y_h = 0 \quad \Rightarrow \lambda = \frac{d}{y_t}$$

So

$$f_s(t,h) = h^{d/y_t} \phi_{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)$$

such that all dependence on t of f_s is absorbed into the function $\phi_\pm$.

Now
$$f_s(t,h) = \begin{cases} h^{d/y_t} \phi_+\left(\frac{h}{t^{\bar{\nu}y_h}}\right), & t>0, \\ h^{d/y_t} \phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right), & t<0 \end{cases}$$

and f_s should be analytic at $t=0$ for any fixed $h \neq 0$.

This means that $\phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right)$ $\left(\phi_+\left(\frac{h}{t^{\nu y_h}}\right)\right)$ is the analytic continuation of $\phi_+\left(\frac{h}{t^{\nu y_h}}\right)$ $\left(\phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right)\right)$ to the region $t<0$ $(t>0)$. If we do analytic continuation of ϕ_+ (ϕ_-) to $t<0$ $(t>0)$, so that both two functions are well-defined inside $t \in (-\varepsilon, \varepsilon)$, $\varepsilon>0$, then clearly

$$\phi_+\left(\frac{h}{t^{\nu y_h}}\right) = \phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right), \quad t \in (-\varepsilon, \varepsilon).$$

Notice that $\phi_+\left(\frac{h}{t^{\nu y_h}}\right)$ and $\phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right)$ are smooth functions of t, but $\phi_+(x)$ and $\phi_-(x)$ themselves may not be smooth with respect to x.

Let's write
$$\phi_+\left(\frac{h}{t^{\nu y_h}}\right) = \phi_+\left(\left(\frac{t}{h^{\frac{1}{\nu y_h}}}\right)^{-\nu y_h}\right) = \mathcal{F}_+\left(\frac{t}{h^{\frac{1}{\nu y_h}}}\right)$$
$$\phi_-\left(\frac{h}{(-t)^{\nu y_h}}\right) = \phi_-\left(\left(\frac{-t}{h^{\frac{1}{\nu y_h}}}\right)^{-\nu y_h}\right) = \mathcal{F}_-\left(-\frac{t}{h^{\frac{1}{\nu y_h}}}\right)$$

Now there exists $\varepsilon_1>0$ and $\varepsilon_2>0$, and $h_0 \neq 0$, s.t.

$$\mathcal{F}_+\left(\frac{t}{h^{\frac{1}{\nu y_h}}}\right) = \mathcal{F}_-\left(-\frac{t}{h^{\frac{1}{\nu y_h}}}\right) = \mathcal{F}(t,h)$$

is smooth in $(-\varepsilon_1, \varepsilon_1) \times (h_0 - \varepsilon_2, h_0+\varepsilon_2)$, since $\mathcal{F}(t,h)$ is analytic at $(t=0, h=h_0)$.

That $\mathcal{F}_\pm\left(\pm \frac{t}{h^{\frac{1}{\nu y_h}}}\right)$ are analytic in $(-\varepsilon_1, \varepsilon_1) \times (h_0-\varepsilon_2, h_0+\varepsilon_2)$ clearly indicates that $\mathcal{F}_\pm(x)$ are analytic in $\left(-\frac{\varepsilon_1}{(2h_0)^{\frac{1}{\nu y_h}}}, \frac{\varepsilon_1}{(2h_0)^{\frac{1}{\nu y_h}}}\right)$.

Hence we can expand $\mathcal{F}_\pm(x)$ in vicinity of $x=0$ as a power series:

$$\mathcal{F}_\pm(x) = \sum_{n=0}^{+\infty} a_n^\pm x^n, \text{ for small enough } x.$$

Then we must have from $\mathcal{F}_+\left(\frac{t}{h^{\frac{1}{\nu y_h}}}\right) = \mathcal{F}_-\left(-\frac{t}{h^{\frac{1}{\nu' y_h}}}\right)$ the following

$$\sum_{n=0}^{+\infty} a_n^+ \left(\frac{t}{h^{\frac{1}{\nu y_h}}}\right)^n = \sum_{n=0}^{+\infty} a_n^- \left(\frac{-t}{h^{\frac{1}{\nu' y_h}}}\right)^n$$

i.e.

$$a_n^+ h^{-\frac{n}{\nu y_h}} = (-1)^n a_n^- h^{-\frac{n}{\nu' y_h}},$$

which holds for a small range of h around h_0. But since both the functions $h^{-\frac{n}{\nu y_h}}$ and $h^{-\frac{n}{\nu' y_h}}$ are analytic at $h_0 \neq 0$, we can establish the fact that

$$a_n^+ = (-1)^n a_n^-$$

$$h^{-\frac{n}{\nu y_h}} = h^{-\frac{n}{\nu' y_h}}, \quad \text{for } h \text{ around } h_0.$$

Clearly, we must have $\nu = \nu'$ to satisfy this.

Problem 2. (Goldenfeld Exercise 9-3)

Solution:

(a) As we've been familiar with, the 1D Ising model can be readily solved by transfer matrix method:

$$Z_N = \mathrm{tr}\, e^{\mathcal{H}} = \mathrm{tr}(T^N)$$

where

$$T = \begin{bmatrix} e^{h+K+K_0} & e^{-K+K_0} \\ e^{-K+K_0} & e^{-h+K+K_0} \end{bmatrix}.$$

Let's then do a partial trace over the even numbered sites:

$$Z_N = \sum_{s_1 \cdots s_N} \langle s_1|T|s_2\rangle\langle s_2|T|s_3\rangle \cdots \langle s_N|T|s_1\rangle$$

$$= \sum_{s_1 s_3 \cdots} \sum_{s_2 s_4 \cdots} \langle s_1|T|s_2\rangle\langle s_2|T|s_3\rangle \cdots \langle s_N|T|s_1\rangle$$

$$= \sum_{s_1 s_3 \cdots} \langle s_1|T^2|s_3\rangle\langle s_3|T^2|s_5\rangle \cdots$$

$$= \mathrm{tr}\left((T^2)^{\frac{N}{2}}\right)$$

But

$$T^2 = \begin{bmatrix} e^{h+2K_0}\cosh(h+2K) & e^{2K_0}\cosh(h) \\ e^{2K_0}\cosh(h) & e^{-h+2K_0}\cosh(-h+2K) \end{bmatrix}$$

and so if we impose

$$T^2 = \begin{bmatrix} e^{h'+K'+K_0'} & e^{-K'+K_0'} \\ e^{-K'+K_0'} & e^{-h'+K'+K_0'} \end{bmatrix}$$

we must have

$$\begin{cases} e^{h+2K_0}\cosh(2K+h) = e^{h'+K'+K_0'} & -① \\ e^{-h+2K_0}\cosh(2K-h) = e^{-h'+K'+K_0'} & -② \\ e^{2K_0}\cosh(h) = e^{-K'+K_0'} & -③ \end{cases}$$

Then

$$\frac{①}{②} \Rightarrow e^{2h'} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$$① \times ② \Rightarrow e^{2K'+2K_0'} = e^{4K_0}\cos(2K+h)\cos(2K-h) \qquad -④$$

$$④ \times ③^2 \Rightarrow e^{4K_0'} = e^{8K_0}\cos(2K+h)\cos(2K-h)\cosh^2(h)$$

$$\frac{④}{③^2} \Rightarrow e^{4K'} = \frac{\cos(2K+h)\cos(2K-h)}{\cosh^2(h)}$$

In summary,

$$\begin{cases} e^{2h'} = e^{2h}\,\dfrac{\cosh(2K+h)}{\cos(2K-h)} \\ e^{4K'} = \dfrac{\cos(2K+h)\cos(2K-h)}{\cosh^2(h)} \\ e^{4K_0'} = e^{8K_0}\cos(2K+h)\cos(2K-h)\cosh^2(h) \end{cases}$$

· The Hamiltonian of 1D Ising model has a time reversal symmetry

$$H(h, K, \{S_i\}) = H(-h, K, \{-S_i\})$$

This implies

$$Z(-h, K, K_0) = Z(h, K, K_0).$$

It is easy to check that under RG operation R, which induces a map on parameter space as defined by solution of the above recursion relation, we have

$$(h', K', K_0') = R(h, K, K_0) \Rightarrow (-h', K', K_0') = R(-h, K, K_0),$$

i.e. (T is time reversal operation)

$$\begin{tikzcd} (h, K, K_0) \arrow[r, mapsto, "R"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowleft"] & (h', K', K_0') \arrow[d, "T"] \\ (-h, K, K_0) \arrow[r, mapsto, "R"] & (-h, K, K_0)' \end{tikzcd}$$

which means ~~that~~ on parameter space R and T commute. So R must preserves time reversal symmetry if there is one in the original problem.

Notice that we've included K_0 in H which does not show up in the usual Ising Hamiltonian. This is because RG will generate constant terms. So even if we start with a model with $K_0 = 0$, non zero K_0 will appear after we do RG. Hence we should include such a term in the flow.

(b) Let's set $h=0$. Then it is easy to read from the second equation of the RG flow equations that

$$e^{4K'} = \cosh^2 2K.$$

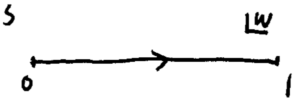
To find fixed points, let

$$e^{4K_*} = \cosh^2 2K_*.$$

Solutions are clearly $K_* = 0$ & $+\infty$. The corresponding $w = e^{-2K} = 1, 0$.

Since $K \sim \frac{1}{T}$, $K_* = 0$ corresponds to $T = +\infty$, the disordered phase; while $K_* = +\infty$ corresponds to $T=0$, the critical point.

Flow in w is



(c) Let's work in vicinity of $T=0$. Here K is very large, so

$$\cosh^2(2K) = \left(\frac{e^{2K}+e^{-2K}}{2}\right)^2 \approx \left(\frac{e^{2K}}{2}\right)^2 = \frac{1}{4}e^{4K}.$$

Hence

$$e^{4K'} = \frac{1}{4}e^{4K}$$

$$\Rightarrow K' = \frac{1}{4}(4K - 2\ln 2) = K - \frac{1}{2}\ln 2$$

This gives

$$\frac{1}{T'} = \frac{1}{T} - \frac{1}{2}\ln 2$$

$$\Rightarrow T' = \frac{T}{1-\frac{1}{2}\ln 2 \cdot T} \approx T + \frac{1}{2}(\ln 2)T^2 .$$

If we linearize the equation, we will get

$$\left.\frac{\partial T'}{\partial T}\right|_{T=0} = 1,$$

which means that $2^{y_t} = 1 \Rightarrow y_t = 0$ (notice $b=2$ in this case).

This does not mean that T doesn't flow. T does not change up to linear order, but it flows to high temperature region to higher orders. The flow around $T=0$ is rather slow.

(d) Let's solve the following for $h \neq 0$:

$$\begin{cases} e^{4K'} = \dfrac{\cosh(2K+h)\cos(2K-h)}{\cosh^2(h)} \\ e^{2h'} = e^{2h}\dfrac{\cosh(2K+h)}{\cosh(2K-h)} \end{cases}$$

If $K=0$, we see that the equations are satisfied for arbitrary h, and $(K'=0,\ h'=h)$. So $(K=0,\ h$ arbitrary$)$ ~~are~~ form a line of critical points.

If $K=+\infty$, first equation is satisfied. For second one,

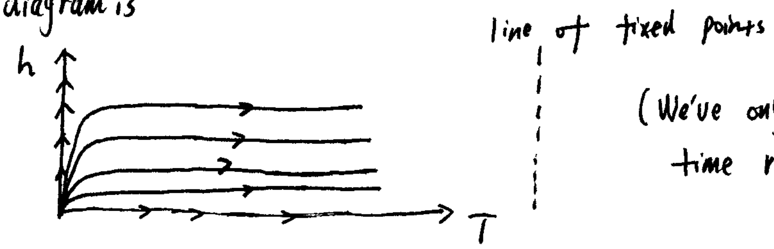
$$\frac{\cosh(2K+h)}{\cosh(2K-h)} \approx \frac{\frac{1}{2}e^{2K+h}}{\frac{1}{2}e^{2K-h}} = e^{2h},$$

so that $e^{2h'} = e^{4h}$, i.e. $h^* = 0$. Hence $(K=+\infty,\ h=0)$ is a fixed point.

These are the only fixed points of the RG recursion relation.

Notice that for very large K, $h'=2h$, so clearly $y_h = 1$.

The flow diagram is



(We've only drawn $h>0$ due to time reversal symmetry)

(e) The results show that except T=0, all finite temperatures flow to high temperature region, hence corresponding to disordered phase. So there is no phase transition at a finite temperature.

We can derive the formula for correlation by exact calculation using transfer matrix. (see eqn. 3.111 in Goldenfeld's book or Professor Wu's lecture notes)

$$\xi = \frac{1}{\ln \coth K}.$$

But
$$\ln \coth K = \ln \frac{e^K + e^{-K}}{e^K - e^{-K}}$$
$$= \ln\left(\frac{1+e^{-2K}}{1-e^{-2K}}\right)$$
$$\simeq \ln(1+e^{-2K})^2 \simeq \ln(1+e^{-4K}) \simeq e^{-4K} \quad (K \to +\infty)$$

Hence $\xi \sim e^{4K} = e^{\frac{4}{T}} \quad (T \to 0)$.

So ξ diverges much quicker than any inverse power law behaviour $(\frac{1}{T})^{\nu}$.
This implies $\nu = +\infty$.

But we've seen in problem 9.1 that $\nu = \frac{1}{y_t}$, hence $y_t = 0$, in consistent with the result from RG method.

To get y_h, we investigate the scaling behaviour of ξ with respect to h. From eqn. 3.109 Goldenfeld's book,

$$\xi = \frac{1}{\log(\lambda_1/\lambda_2)},$$

where

$$\lambda_{1,2} = e^K\left[\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4K}}\right]$$

Hence if $K \to +\infty$,

$$\lambda_1/\lambda_2 \simeq \frac{\cosh(h)+\sinh(h)}{\cosh(h)-\sinh(h)} \simeq e^{2h} \simeq 1+2h \quad (h \to 0)$$

Thus $\log(\lambda_1/\lambda_2) \simeq 2h, \quad (h \to 0, K \to +\infty)$

So

$$\xi \sim \frac{1}{h}.$$

Clearly $y_h = 1$, consistent with RG result.

Problem 3. (Goldenfeld Exercise 12-3)

Solution :

(a) These two RG flow equations have already been derived in Professor Congjun Wu's lecture notes. The constants A and B take different values for different symmetries. They are all positive anyway, since critical value of r will be shifted downward with respect to its mean-field value, and u has a physical fixed point value for $\varepsilon > 0$.

(b) This part of the problem has also been discussed in Professor Wu's lectures. There are two fixed points, one is Gaussian fixed point with $r^* = u^* = 0$, while the other is Wilson-Fisher fixed point with $\left(r^* = -\frac{1}{2}\frac{A}{B}\varepsilon,\ u^* = \frac{\varepsilon}{B}\right)$.

If we linearize the RG equation around fixed point, we will have

$$\begin{bmatrix} \frac{d\,\delta r}{ds} \\ \frac{d\,\delta u}{ds} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r}\left(\frac{dr}{ds}\right) & \frac{\partial}{\partial u}\left(\frac{dr}{ds}\right) \\ \frac{\partial}{\partial r}\left(\frac{du}{ds}\right) & \frac{\partial}{\partial u}\left(\frac{du}{ds}\right) \end{bmatrix}_{(r,u)=(r^*,u^*)} \begin{bmatrix} \delta r \\ \delta u \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \frac{Au^*}{(1+r^*)^2} & \frac{A}{1+r^*} \\ 2B\frac{u^{*2}}{(1+r^*)^3} & \varepsilon - \frac{2Bu^*}{(1+r^*)^2} \end{bmatrix} \begin{bmatrix} \delta r \\ \delta u \end{bmatrix}$$

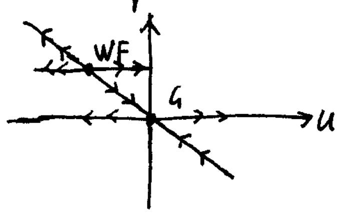
We can then get the familiar results

$$\begin{bmatrix} \frac{d\,\delta r}{ds} \\ \frac{d\,\delta u}{ds} \end{bmatrix}_{\text{Gaussian}} = \left(\begin{bmatrix} 2 & A \\ 0 & \varepsilon \end{bmatrix} + O(\varepsilon^2) \right) \begin{bmatrix} \delta r \\ \delta u \end{bmatrix}_{\text{Gaussian}}$$

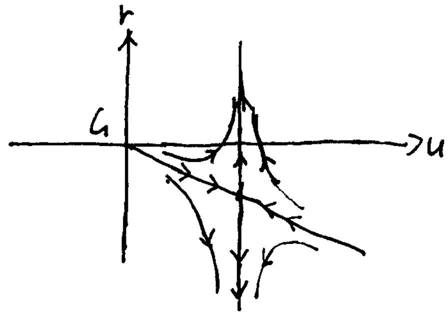
$$\begin{bmatrix} \frac{d\,\delta r}{ds} \\ \frac{d\,\delta u}{ds} \end{bmatrix}_{W\text{-}F} = \left(\begin{bmatrix} 2 - \frac{A}{B}\varepsilon & A\left(1 + \frac{1}{2}\frac{A}{B}\varepsilon\right) \\ 0 & -\varepsilon \end{bmatrix} + O(\varepsilon^2) \right) \begin{bmatrix} \delta r \\ \delta u \end{bmatrix}_{W\text{-}F}$$

The flows are

$\varepsilon < 0$



$\varepsilon > 0$



The temperature-like scaling quantity must be zero along the critical line. Hence $t = r + \frac{1}{2}Au$ is just such a quantity, and actually it is just the first two terms of the temperature derived in Professor Wu's lecture notes.

(c) We need the flow equation for the above defined t. We set $\varepsilon = 0$.

$$\frac{dt}{ds} = \frac{dr}{ds} + \frac{1}{2}A\frac{du}{ds}$$
$$= 2r + \frac{Au}{1+r} - \frac{1}{2}ABu^2$$

At $d=4$, coupling u is asymptotic free in the infrared, so we only keep terms linear in u, since remember that we are lowering the cutoff and u actually decays.

Then $\frac{dt}{ds} = 2r + Au - Aur$.

But notice that

$$(2-Au)t = (2-Au)(r + \frac{A}{2}u)$$
$$= 2r + Au - Aur - \frac{1}{2}A^2u^2,$$

We see that up to $O(u)$,

$$\frac{dt}{ds} = (2-Au)t .$$

This equation is very easy to solve:

$$\ln\frac{t}{t_0} = \int_0^s (2-Au)ds'$$

i.e.

$$t(s) = t_0\, e^{\int_0^s (2-Au(s'))ds'} .$$

To get the form of $t(s)$, we need that of $u(s)$, which is easy:

$$\frac{du}{ds} = -Bu^2$$
$$\Rightarrow d\left(\frac{1}{u}\right) = B\,ds$$
$$\Rightarrow \frac{1}{u} - \frac{1}{u_0} = Bs$$
$$\Rightarrow u(s) = \frac{1}{\frac{1}{u_0} + Bs} = \frac{u_0}{1+u_0Bs} .$$

Thus
$$t(s) = t_0\, e^{\int_0^s \left(2 - \frac{Au_0}{1+u_0Bs}\right)ds}$$
$$= t_0\, e^{2s} \cdot e^{-\int_0^s \frac{Au_0\,ds}{1+u_0Bs}}$$

in which

$$Au_0\int_0^s \frac{1}{1+u_0Bs}ds = \frac{A}{B}\ln(1+u_0Bs) \sim \frac{A}{B}\ln(u_0Bs) \quad \text{if } s \gg 1.$$

In summary,

$$t(s) = t_0(u_0B)^{-\frac{A}{B}}\, e^{2s} \cdot s^{-\frac{A}{B}} \sim t_0 l^2 (\log l)^{-\frac{A}{B}} \quad (\text{since } s = \ln l)$$

To get relation between ℓ and t, consider we drive the system to $|t(s)|=1$, and write $t_0=t$. then

$$\pm 1 \sim t\,\ell^2(\log \ell)^{-\frac{A}{B}}.$$

Hence

$$\ell^2 \sim \frac{1}{|t|}(\log \ell)^{\frac{A}{B}}$$

$$\Rightarrow \ell \sim |t|^{-1/2}(\log \ell)^{\frac{A}{2B}}.$$

If we are only interested in leading scaling behaviour of ℓ, we can set $\ell=|t|^{-1/2}$ in $\log \ell$, so that

$$\ell \sim |t|^{-1/2}\left(-\frac{1}{2}\log|t|\right)^{\frac{A}{2B}}.$$

Dropping irrelevant constants, we get

$$\ell \sim |t|^{-1/2}(-\log|t|)^{\frac{A}{2B}}$$

For Ising universality class, $\frac{A}{B}=\frac{1}{3}$, and for $t>0$,

$$\ell \sim t^{-1/2}(-\log t)^{\frac{1}{6}}.$$

Relation between ξ and ℓ is straightforward:

$$\xi(t_0) = \ell\,\xi(t(s)).$$

Again set $t_0=t$ and $|t(s)|=1$, we see

$$\xi(t) = \ell\,\xi(\pm 1) \sim \ell.$$

So

$$\xi \sim t^{-1/2}(-\log t)^{\frac{1}{6}} \quad (t>0).$$

(d) From a naive scaling form of free energy density that we've been familiar with we get

$$f_s(t,u) = \ell^{-d} f_s(t_\ell, u_\ell)$$

If ℓ is very large, u_ℓ approaches u_* so that

$$f_s(t,u) = \ell^{-d} f_s(t_\ell, u_*).$$

We've seen in part (c) that if we scale t to $t_\ell=\pm 1$, the scaling factor that we need is $\ell \sim t^{-1/2}(-\log t)^{\frac{A}{2B}} = t^{-1/2}(-\log t)^{\frac{1}{6}}$ $(t>0)$.

Plugging this into f we get (set $d=4$)

$$f_s(t,u) \sim t^2(-\log t)^{-2/3}.$$

Specific heat $\sim \frac{\partial^2 f}{\partial t^2} \sim (-\log t)^{-2/3}$ to leading order singularity.

But this approach can in principle be wrong, and actually it is. The reason is that while we get the flow of field theories in coupling constant space, we've dropped a lot of constant terms in the action, for example, Gaussian integral of free part of the fast modes, and change of functional measure when we rescale the field ϕ.

These constants actually do not appear and cancel between numerator and denominator if we calculate correlation functions. But they in principle should be kept if we are interested in free energy. When we do RG many times (i.e. when ℓ is very large), the accumulation of these terms may contribute to singular behaviours which has to be taken into account in f_s.