Assignment #2

PHSY 217

Renormalization Group

Problem 1. (Goldenfeld Exercise 5-1)

Solution:

(a) We briefly recall some key results in exercise 3-3.

The action is

$$S = \frac{1}{2}\sum_{ij}(\psi_i - H_i)(J+\lambda I_N)^{-1}_{ij}(\psi_j - H_j) - \frac{1}{\beta}\sum_i \ln(2\cosh(\beta\psi_i))$$

in which we've added to J a constant matrix λI_N as explained in 3-3 (b) to ensure positive-definiteness.

By extremizing S we get the saddle point solution of $\bar{\psi}_i$:

$$\sum_j \tilde{J}^{-1}_{ij}(\bar{\psi}_j - H_j) = \tanh\bar{\psi}_i \,, \quad (\tilde{J} = J + \lambda I_N)$$

then using $m_i = -\frac{\partial F}{\partial H_i}$ we get

$$m_i = \tanh\beta\psi_i \quad \left(= \sum_j \tilde{J}^{-1}_{ij}(\bar{\psi}_j - H_j)\right)$$

from which equation of state can be read

$$H_i = \frac{1}{\beta}\operatorname{arctanh} m_i - \sum_j (J+\lambda I_N)_{ij}\, m_j .$$

Notice that the result for equation of state depends on the constant matrix λI_N that we add. This is because although $\sum_i J_{ii} S_i^2$ is a constant since $S_i^2 = 1$, we lose such a constraint when finding the extremal solution, or equivalently replacing S_i^2 with $S_i\langle S_i\rangle$ when doing mean field theory.

But the true mean field theory results correspond to $\lambda = 0$. To cure this discrepancy, imagine that we can first do the above steps for large enough $\lambda > 0$, then do an analytic continuation for the final results and set $\lambda = 0$.

Now set $m_i \equiv m$. Then

$$\bar{S}(m) = \frac{1}{2}\sum_{ij} J_{ij} m_i m_j - \frac{1}{\beta}\sum_i \log\frac{2}{\sqrt{1-m_i^2}}$$

$$= N\left(dJ m^2 - \frac{1}{\beta}\log\frac{2}{\sqrt{1-m^2}}\right)$$

~~Clearly~~

By definition of Gibbs free energy, we have

$$\Gamma(m) = \bar{S}(m) + NHm,$$

so

$$\frac{1}{N}\Gamma(m) = \frac{1}{N}\bar{S}(m) + \left(\frac{1}{\beta}\operatorname{arctanh} m - (2dJ+\lambda)m\right)\cdot m$$

$$= (dJ + \tfrac{1}{2}\lambda)m - \frac{1}{\beta}\log\frac{2}{\sqrt{1-m^2}} + \frac{1}{\beta}m\cdot\operatorname{arctanh} m \quad -(2dJ+\lambda)m^2 .$$

By the standard expansion formulas,

$$\operatorname{arctanh} x = x + \frac{1}{3}x^3 + O(x^4)$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$$

$$\log(1+x) = x + \frac{1}{2}x^2 + O(x^3)$$

It is straightforward to get the following expansion of $\Gamma(m)$ around $m=0$:

$$\frac{1}{N}\Gamma(m) = \frac{1}{\beta}\cdot\left\{\frac{1}{2}\left[1-(2dJ+\lambda)\beta\right]m^2 + \frac{1}{12}m^4 - \log 2\right\} + O(m^6)$$

As mentioned above, now we can set $\lambda=0$ by virtue of analytic continuation to recover the familiar mean field theory results:

$$\frac{1}{N}\Gamma(m) = \frac{1}{\beta}\left(\frac{1}{2}(1-2dJ\beta)m^2 + \frac{1}{12}m^4 - \log 2\right) + O(m^6)$$

The equation of state is $H = \frac{1}{N}\frac{\partial\Gamma(m)}{\partial m}$. Hence magnetization at zero field is determined by extremal point of $\Gamma(m)$: $\frac{\partial\Gamma(m)}{\partial m} = 0$, and spontaneous magnetization develops when this equation has non-zero solution. Clearly this holds when coefficient of quadratic term becomes negative, i.e. $2dJ\beta_c = 1$, and this gives $T_c = 2dJ/k_B$.

Now let's calculate β. For this let

$$0 = H = \frac{1}{N}\frac{\partial\Gamma(m)}{\partial m} = \left(1-\frac{\beta}{\beta_c}\right)m + \frac{1}{3}m^3,$$

where $1-\frac{\beta}{\beta_c} = 1-\frac{T_c}{T} = t + O(t^2)$.

To get β, we consider $t<0$ the ordered phase, and it is easy to see that apart from the trivial solution of $m=0$, there are two non zero solutions as $m = \pm\sqrt{-3t}$, and so by definition $\beta = 1/2$.

Then let's do δ. For this we set $T = T_c$ so that $H = \frac{1}{3}m^3$, and so $m = (3H)^{\frac{1}{3}}$. Clearly $\delta = 3$.

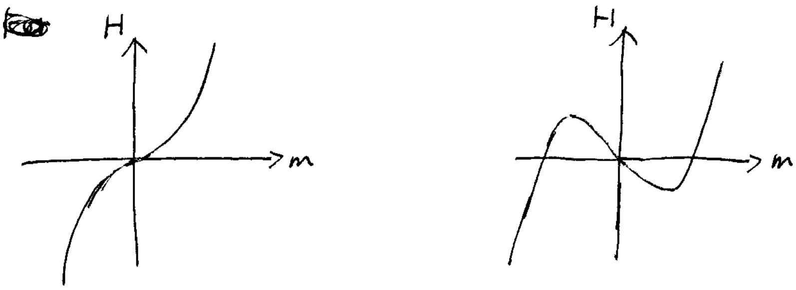
(b) Relation between H and m is given by equation of state

$$H = \frac{1}{\beta}\operatorname{arctanh} m - \frac{1}{\beta_c}m.$$

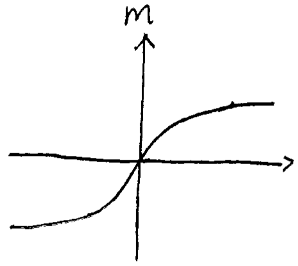
So

$$\beta_c H = (1+t)\left(m+\frac{1}{3}m^3\right) - m + \text{higher order terms}$$
$$= tm + \frac{1}{3}m^3 + \text{higher order terms}.$$

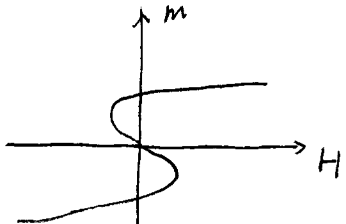
Let's sketch the relation $\beta_c H = tm + \frac{1}{3}m^3$



$T > T_c$ $\qquad\qquad$ $T < T_c$



$T > T_c$



$T < T_c$

Clearly one can see that when $T<T_c$ a single H corresponds to three values of m and so it is an unphysical region.

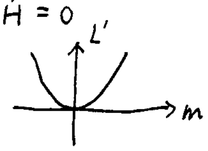
Now we define "Landau free energy"

$$L'(m,\hat{H}) = \Gamma(m) - N m \hat{H}$$

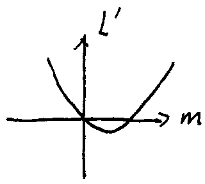
$$\frac{1}{N}L'(m,\hat{H}) = \frac{1}{\beta_c}\left(\frac{1}{2}tm^2 + \frac{1}{12}m^4 - m\cdot\beta_c\hat{H}\right)$$

We sketch $\frac{1}{N}L'(m,\hat{H})$ for different cases as follows:

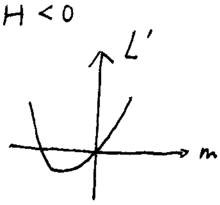
$t>0$ ① $\hat{H}=0$



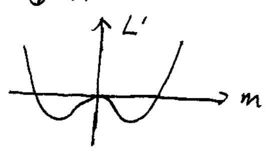
② $\hat{H}>0$



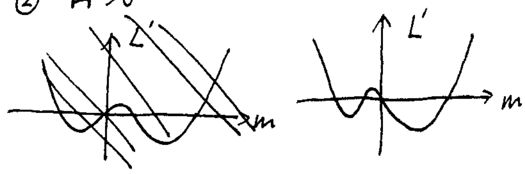
③ $\hat{H}<0$



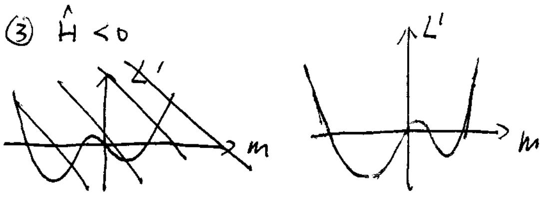
$t<0$ ① $\hat{H}=0$



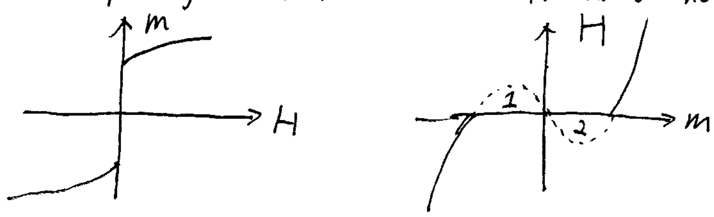
② $\hat{H}>0$



③ $\hat{H}<0$



From these figures we see that the condition that L' be minimized for fixed $\hat{H}$ really removes the unphysical portion of the curve since now there is only one value of m corresponding to an $\hat{H}$. The $m-H$ curve now becomes



This is really Maxwell's equi-area construction since area 1 = area 2.

Notice that minimization of L' gives m is as expected, since value of $L'(m,\hat{H})$ at extremal point is nothing but the free energy in Exercise 3-3 (e). (in Γ we've added NmH and now we substract it from Γ, i.e. the $-mN\hat{H}$ term). So we are essentially selecting out the m (or equivalently $\bar{\Psi}$) for fixed H when there are more than one saddle solutions.

Problem 2. (Goldenfeld Exercise 6-3)

Solution:

(a) For purpose of doing Gaussian fluctuations we expand the action to second order of $\delta\varphi_i$ at the saddle point $\bar{\varphi}_i$. For this we need to calculate $\left.\frac{\partial^2 S}{\partial\varphi_m \partial\varphi_n}\right|_{\varphi=\bar{\varphi}}$.

It is pretty straightforward to get

$$\frac{\partial^2 S}{\partial\varphi_m \partial\varphi_n} = \tilde{J}^{-1}_{mn} - \beta(1-\tanh^2\beta\bar{\varphi}_n)\,\delta_{nm}$$

After doing the functional integral we get $\left(\det \frac{\partial^2 S}{\partial\varphi_m\partial\varphi_n}\right)^{-1/2}$ with constants like $\beta^{-\frac{N}{2}}$ $(\sqrt{2\pi})^N$ dropped.

Notice that we can take out a matrix $\tilde{J}^{-1}$ from $\frac{\partial^2 S}{\partial\varphi_m\partial\varphi_n}$, and get the free energy to be

$$F = S(\bar{\varphi}_i) + \frac{1}{2\beta}\log\det\left(\delta_{ij} - \beta(1-\tanh^2\beta\,\bar{\varphi}_i)\tilde{J}_{ij}\right) + \text{constant}$$

where we've absorbed $-\frac{1}{\beta}\log\left(\left(\det\tilde{J}^{-1}\right)^{-1/2}\right)$ into the constant term.

So we can just take free energy to be

$$F = S(\bar{\varphi}_i) + \frac{1}{2\beta}\log\det\left(\delta_{ij} - \beta(1-\tanh^2\beta\,\bar{\varphi}_i)\,\tilde{J}_{ij}\right).$$

(b) Let's write $F = S(\bar{\varphi}_i, H_i) + \varepsilon\,\delta\Gamma$. Finally we will take $\varepsilon = 1$.

The equation of state is

$$m_i = -\frac{\partial F}{\partial H_i} = -\frac{\partial S}{\partial H_i} - \varepsilon\frac{\partial\delta\Gamma}{\partial H_i} = \bar{m}_i - \varepsilon\frac{\partial\delta\Gamma}{\partial H_i}$$

(where we've denote $\bar{m}_i$ to be the mean field magnetization)

hence $\bar{m}_i = m_i + \varepsilon\frac{\partial\delta\Gamma}{\partial H_i}$.

In principle we can get the relation between H_j and m_i from this equation.

By definition

$$\begin{aligned}
\Gamma(m_i) &= F(m_i) + \sum_i H_i m_i \\
&= \bar{S}(\bar{m}_i) + \varepsilon\,\delta\Gamma + \sum_i H_i m_i \\
&= \bar{S}(\bar{m}_i) + \varepsilon\,\delta\Gamma + \sum_i H_i\left(\bar{m}_i - \varepsilon\frac{\partial\delta\Gamma}{\partial H_i}\right) \\
&= \bar{\Gamma}(\bar{m}_i) + \varepsilon\,\delta\Gamma - \varepsilon\sum_i H_i\frac{\partial\delta\Gamma}{\partial H_i} \\
&= \bar{\Gamma}\left(m_i + \varepsilon\frac{\partial\delta\Gamma}{\partial H_i}\right) + \varepsilon\,\delta\Gamma - \varepsilon\sum_i H_i\frac{\partial\delta\Gamma}{\partial H_i} \\
&= \bar{\Gamma}(m_i) + \varepsilon\sum_i\frac{\partial\Gamma}{\partial m_i}\frac{\partial\delta\Gamma}{\partial H_i} + O(\varepsilon^2) + \varepsilon\,\delta\Gamma - \varepsilon\sum_i H_i\frac{\partial\delta\Gamma}{\partial H_i}
\end{aligned}$$

So up to $O(\varepsilon)$ we get

$$\Gamma(m_i) = \bar{\Gamma}(m_i) + \varepsilon\,\delta\Gamma$$

where $\bar{\Gamma}$ denotes the mean field function.

(c) To get static magnetic susceptibility, we set $m_i \equiv m$, and then $\Gamma(m) = \bar{\Gamma}(m) + \delta\Gamma$.

Then

$$\chi_T^{-1} = \frac{\partial H}{\partial m}\Big|_{m=0} = \frac{\partial^2 \Gamma}{\partial m^2}\Big|_{m=0} = \frac{\partial^2 \bar{\Gamma}(m)}{\partial m^2}\Big|_{m=0} + \frac{\partial^2 \delta\Gamma}{\partial m^2}\Big|_{m=0}$$

where we've set $m=0$ since we consider the disordered phase, and notice that $\frac{\partial^2 \bar{\Gamma}}{\partial m^2}\big|_{m=0}$ is the mean field $\bar{\chi}_T^{-1} \sim t$.

Hence

$$\chi_T^{-1} = At + \frac{\partial^2 \delta\Gamma}{\partial m^2}\Big|_{m=0} .$$

Then χ_T^{-1} vanishes at $t = -\frac{1}{A}\frac{\partial^2 \delta\Gamma}{\partial m^2}\big|_{m=0}$.

One can check that in mean field theory $\bar{\chi}_T^{-1} = 2k_B T_c t$, so $A = 2k_B T_c > 0$. Then if we can prove that $\frac{\partial^2 \delta\Gamma}{\partial m^2}\big|_{m=0} > 0$, we will obtain the conclusion that critical temperature is shifed downward.

For a constant configuration $m_i \equiv m$, $\Psi_i \equiv \Psi$ is also a constant in space. So

$$\frac{\partial^2}{\partial m^2}\delta\Gamma\Big|_{m=0} = \frac{\partial^2}{\partial m^2}\frac{1}{2\beta}\log\det\left[\delta_{ij} - \beta(1-\tanh^2(\beta\Psi))\tilde{J}_{ij}\right]\Big|_{m=0}$$

$$= \frac{\partial^2}{\partial m^2}\frac{1}{2\beta}\,\mathrm{tr}\log\left[I_N - \beta(1-\tanh^2\beta\Psi)\tilde{J}\right]\Big|_{m=0}$$

Notice that $\bar{m} = \tanh\beta\Psi \approx \beta\Psi$ when $\bar{m}$ is small, and $\bar{m} = m$ up to $O(\varepsilon^0)$.
Hence we can approximate the above expression as

$$\frac{\partial^2}{\partial m^2}\delta\Gamma\Big|_{m=0} = \frac{\partial^2}{\partial m^2}\left[\frac{1}{2\beta}\,\mathrm{tr}\log(I_N - \beta(1-m^2)\tilde{J})\right]\Big|_{m=0}$$

$$= \frac{1}{2\beta_c}\frac{\partial^2}{\partial m^2}\left[\mathrm{tr}\log(I_N - (1-m^2)\beta_c\tilde{J})\right]\Big|_{m=0}$$

where we've approximated β as β_c.

Notice that $\beta_c = \frac{1}{2dJ}$, and if the ~~norm o~~ matrix norm of $\beta_c\tilde{J}$ is smaller than that of I_N, or equivalently if all eigenvalues of $\beta_c\tilde{J}$ is strictly smaller than 1, we can do an expansion of log-function. Actually this holds for matrix J with no constant matrix added, as can be seen from a straightforward diagonalization of the hopping matrix J using translational invariance. ~~But really we need $\tilde{J}$ to be positive definite, and suppose we can do expansion.~~
~~Then Suppose the expansion can be done, then~~

$$\frac{\partial^2}{\partial m^2}\delta\Gamma\Big|_{m=0} = \frac{1}{2\beta_c}\,\mathrm{tr}\,\frac{\partial^2}{\partial m^2}\left(\sum_{n=1}^{+\infty}(-1)\cdot\frac{1}{n}\cdot(1-m^2)^n(\beta_c\tilde{J})^n\right)\Big|_{m=0}$$

$$= \frac{1}{2\beta_c}\,\mathrm{tr}\sum_{n=1}^{+\infty}\frac{-1}{n}(\beta_c\tilde{J})^n\cdot(-2n)$$

$$= \frac{1}{\beta_c}\sum_{n=1}^{+\infty}\mathrm{tr}(\beta_c\tilde{J})^n$$

Take $\tilde{J} = J$, then by the above comment the expansion is legitimate.
Since eigenvalues of the hopping matrix J appear in pairs $\pm E(\vec{k})$, we see that $\mathrm{tr}(\beta_c J)^n$ vanishes for n to be odd, and positive for n to be even.
So
$$\frac{\partial^2}{\partial m^2} \delta T \Big|_{m=0} = \frac{1}{\beta_c} \sum_{k=0}^{+\infty} \mathrm{tr}\,(\beta_c J)^{2k} > 0 .$$

This shows that the critical temperature will be shifted downward. A final comment is that we take $\tilde{J} = J$ by virtue of analytic continuation mentioned before.

Problem 3. (Goldenfeld Exercise 7-1)

Solution:

(a) Let's set $\begin{cases} x = \xi x' \\ \phi = \xi^{1-d/2}\phi' \end{cases}$ $\left(r_0 = \frac{1}{\xi^2} \right)$, then

$$L = \int d^d x' \left[\frac{1}{2}(\nabla'\phi')^2 + \frac{1}{2}\phi'^2 + \frac{u_n}{n!}\xi^{(1-\frac{n}{2})d+n}\phi'^n \right]$$

So upper critical dimension is

$$\left(1-\frac{n}{2}\right)d_c + n = 0,$$

i.e. $d_c = \frac{n}{n/2-1} = \frac{2n}{n-2}$

(b) Let $n=6$, we find $d_c = 3$.

Hence the critical exponents obtained in 5-2 are accurate when $d>3$, not reliable when $d<3$. While for $d=3$ u_6 is marginal by naive dimensional analysis and mean field theory may not be correct.

(c) Coefficient of $(\nabla\phi)^n$ $(n>2)$ aquires $\xi^{-\frac{d}{2}\cdot(n-2)}$ after rescaling, hence irrelevant. If we add more derivatives, then each derivative contributes a ξ^{-1}, hence irrelevant when $T\to T_c$.