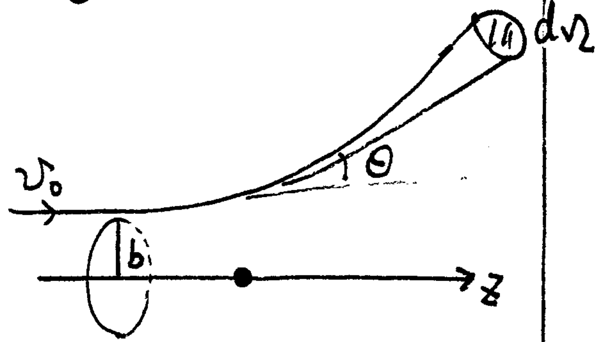
Lect 23. Description of Scattering theory

§ Cross section: classical theory



$$dn = j_i \sigma d\Omega \Rightarrow \sigma = \frac{1}{j_i}\frac{dn}{d\Omega}$$

$$\sigma_t = \int d\Omega\, \sigma(\theta,\varphi) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi\, \sigma(\theta,\varphi)$$

The deflection angle θ depends on the distance b, let's set $b \to b + db$, $\theta \to \theta + d\theta$. Then $dn = j_i\, b db\, d\varphi = j_i \sigma \sin\theta d\theta d\varphi$

$$\Rightarrow \sigma(\theta,\varphi) = \frac{b db}{\sin\theta d\theta}$$

• example Conlumb potential

$$V(r) = \frac{\chi}{r}, \quad \text{and } \chi > 0.$$

From classic physics, the Solution of the trajectory of a particle in the polar coordinate. The force center is the focus

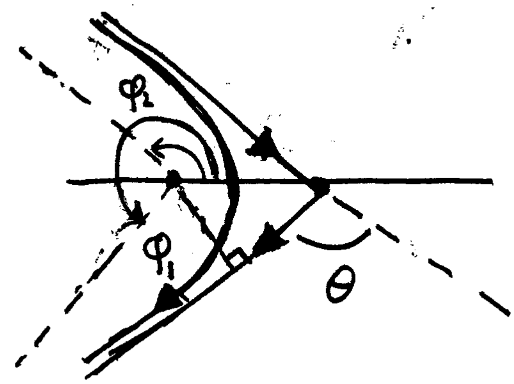
$$r = \frac{P}{1 + e\cos\varphi}, \quad \text{where } e = \sqrt{1 + \frac{2EL^2}{\chi^2 m}} > 1, \text{ eccentricity}$$

$P = \dfrac{L^2}{\chi m}$ is the distance from the focus to the line of directrix

The direction of the asyptotes

$1+e\cos\varphi=0$

$$\begin{cases}\varphi_1 = \pi - \cos^{-1} 1/e \\ \varphi_2 = \pi + \cos^{-1} 1/e\end{cases}$$



The deflection angle $\theta = \pi - 2\cos^{-1} 1/e$

The distance from the focus to the incoming asymptote $m v_0 b = L$

$$\sin\frac{\theta}{2} = \frac{1}{e} \Rightarrow \mathrm{ctg}\frac{\theta}{2} = \sqrt{e^2-1} = \sqrt{\frac{2E}{\kappa^2 m}}\cdot L = \frac{v_0}{\kappa} m v_0 b$$

$$\Rightarrow b = \frac{\kappa}{m v_0^2}\,\mathrm{ctg}\frac{\theta}{2}$$

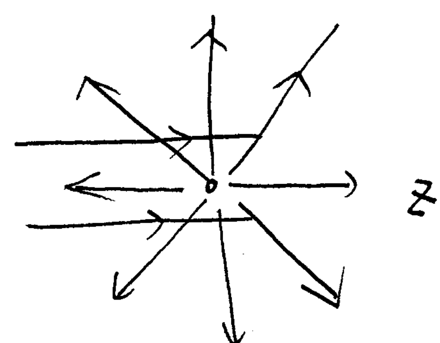
$$\sigma = \frac{b\,db}{\sin\theta\, d\theta} = \left(\frac{\kappa}{m v_0^2}\right)^2 \frac{\mathrm{ctg}\frac{\theta}{2}}{\sin\theta}\,\frac{1/2}{\sin^2\frac{\theta}{2}} = \boxed{\frac{\kappa^2}{16E^2}\,\frac{1}{\sin^4\theta/2} = \sigma}$$

Rutherford formula

§ Quantum mechanics description

incoming wave $\psi_i = e^{ikz}$

scattering wave $\frac{f(\theta)}{r} e^{ikr}$



no dependence on the azimuther angle on φ due to the cylindrical symmetry.

let's consider short range scattering (Coulumb scattering actually doesn't fit into this category). As $r \to \infty$, we have

$$\boxed{\psi \xrightarrow{r\to\infty} e^{ikz} + f(\theta)\frac{1}{r}e^{ikr}} \leftarrow \text{boundary condition}$$

scattering amplitude to be determined

Suppose we have already the solution $f(\theta)$ by solving Schrödinger Eq.

Then
$$j_{in} = \psi_{in}^* \frac{-i\hbar}{2m}\nabla\psi_{in} - c.c. = \frac{\hbar k}{m}$$

$$j_s = \frac{-i\hbar}{2m}\{f^*(\theta)\frac{e^{-ikr}}{r}\frac{\partial}{\partial r}[f(\theta)\frac{e^{ikr}}{r}] - c.c.\} = \frac{\hbar k}{m}|f(\theta)|^2\frac{1}{r^2}$$

$$dn = j_s r^2 d\Omega = j_{in}\sigma\, d\Omega \Rightarrow \frac{\hbar k}{m}|f(\theta)|^2 = \frac{\hbar k}{m}\sigma(\theta)$$

$$\boxed{\sigma(\theta) = |f(\theta)|^2}, \boxed{\sigma_{tot} = \int d\Omega\, |f(\theta)|^2}.$$

We have neglected the interfence between the incoming and scattering waves. Now we need to justify this. Plug $\psi(r) = e^{ikr\cos\theta} + \frac{f(\theta)}{r}e^{ikr}$

into
$$\begin{cases} j = \frac{-i\hbar}{2m}(\psi^*\nabla\psi - c.c.) \\ \nabla_r = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \end{cases}$$

$$\Rightarrow j_\varphi = 0$$

$$j_r = \frac{\hbar k}{m}[\cos\theta + \frac{1}{r^2}|f|^2] + \frac{\hbar}{2m}\{f(\theta)[kr(1+\cos\theta)+i]\frac{e^{ik(r-z)}}{r^2} + c.c.\}$$

$$j_\theta = -\frac{\hbar k}{m}\sin\theta + \frac{\hbar}{2im}[(\frac{df}{d\theta} - ikrf\sin\theta)\frac{1}{r^2}e^{ik(r-z)} - c.c.]$$

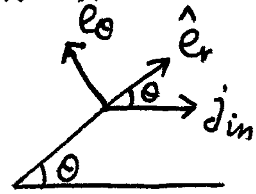
$$+\frac{\hbar}{2im}\left(\frac{df}{d\theta}f^* - f\frac{df^*}{d\theta}\right)/r^3$$

① $1/r^3$ term can be neglected as $r \to \infty$.

② the inteference term is fast oscilating: $\underbrace{e^{ik(r-z)} = e^{ikr(1-\cos\theta)}}_{\text{at } kr \to \infty}$.

unless at $\theta \to 0°$.

$$\Rightarrow \quad j_\varphi = 0, \quad j_r = \frac{\hbar k}{m}\Big(\underbrace{\cos\theta}_{\text{incident wave}} + \underbrace{\frac{|f|^2}{r^2}}_{\text{scattering wave}}\Big) \quad \underbrace{j_\theta = \frac{\hbar k}{m}(-\sin\theta)}_{\text{incident wave}}$$



- Optical theorem

let us consider a sphere with $r \to \infty$, then the net flux is zero due to particle number conservation $\oint j_r r^2 d\Omega = 0$, plug in the expression of $j_r \Rightarrow$

$$\oint |f|^2 d\Omega + \oint \frac{\hbar}{2m} f(\theta)\left[kr(1+\cos\theta)+i\right] e^{ikr(1-\cos\theta)} + c.c.\} = 0$$

Note the result

$$\lim_{kr\to\infty} e^{ikr(1-\cos\theta)} = \frac{2i}{kr}\delta(1-\cos\theta) \quad \text{under the integral } \int d\Omega$$

Ex: please prove it.

$$\Rightarrow \oint \frac{\hbar}{2m} f(\theta)\left[kr(1+\cos\theta)+i\right]\frac{2i}{kr}\delta(1-\cos\theta) \quad + c.c$$

$$= \oint \frac{\hbar}{2m} \left[\frac{2i}{k^2} f(\theta)(1+\cos\theta) + c.c. \right] \delta(1-\cos\theta) + c.c.$$

$$\Rightarrow \oint |f|^2 d\Omega = \frac{-1}{k^2} \int_{-1}^{1} d\cos\theta \int d\varphi \; [2i f(\theta) + c.c.] \, \delta(1-\cos\theta)$$

$$\int_{-1}^{1} d\cos\theta \; \delta(1-\cos\theta) = \frac{1}{2}$$

$$\Rightarrow \oint |f|^2 d\Omega = \frac{2\pi \, 2 \, \mathrm{Im} f(0)}{k^2} = \boxed{\frac{4\pi}{k^2} \mathrm{Im}\, f(0). = \sigma_{tot}}$$