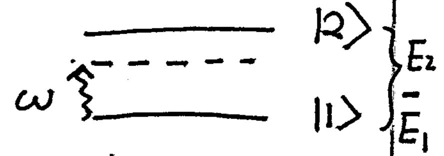
Lect 21 Time-dependent problem — 2 energy levels

§ Two energy level coupled to AC field



$$H = H_0 + V(t)$$

$$H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|, \text{ and } V(t) = \gamma e^{i\omega t} |1\rangle\langle 2| + \gamma e^{-i\omega t} |2\rangle\langle 1|$$

$|1\rangle$, and $|2\rangle$ are energy levels, $V(t)$ can be considered as a driven laser with a frequency ω: $\gamma = \langle 1| e\vec{r}\cdot\vec{E}|2\rangle$. If the coupling matrix element is large, or $E_2 - E_1 \simeq \omega$, we cannot use the linear order pertubation theory in last lecture. For the case of 2-energ levels we can solve it exactly.

$$|\psi(t)\rangle = C_1(t) e^{-iE_1t/\hbar}|1\rangle + C_2(t) e^{-iE_2t/\hbar}|2\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}C_1(t) e^{-iE_1t/\hbar}|1\rangle + i\hbar\frac{\partial}{\partial t}C_2(t) e^{-iE_2t/\hbar}|2\rangle$$
$$+ E_1 C_1(t) e^{-iE_1t/\hbar}|1\rangle + E_2 C_2(t) e^{-iE_2t/\hbar}|2\rangle$$

$$[H_0 + V(t)]|\psi(t)\rangle = C_1(t) E_1 e^{-iE_1t/\hbar}|1\rangle + C_2(t) E_2 e^{-iE_2t/\hbar}|2\rangle$$
$$+ C_1(t) V(t) e^{-iE_1t/\hbar}|1\rangle + C_2(t) V(t) e^{-iE_2t/\hbar}|2\rangle$$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}C_1(t) e^{-iE_1t/\hbar}|1\rangle + i\hbar\frac{\partial}{\partial t}C_2(t) e^{-iE_2t/\hbar}|2\rangle$$
$$= C_1(t) V(t) e^{-iE_1t/\hbar}|1\rangle + C_2(t) V(t) e^{-iE_2t/\hbar}|2\rangle$$
$$= C_1(t)\,\gamma\, e^{-i\omega t - iE_1t/\hbar}|2\rangle + C_2(t)\,\gamma\, e^{i\omega t - iE_2t/\hbar}|1\rangle$$

$$\begin{cases} i\hbar \frac{\partial}{\partial t} C_1(t) = \gamma\, e^{i[\omega - \frac{(E_2-E_1)}{\hbar}]t} C_2(t) = \gamma\, e^{-i\Delta\omega t} C_2(t) \\ i\hbar \frac{\partial}{\partial t} C_2(t) = \gamma\, e^{-i(\omega - \frac{(E_2-E_1)}{\hbar})t} C_1(t) = \gamma\, e^{i\Delta\omega t} C_1(t) \end{cases}$$

$$\Delta\omega = \frac{E_2 - E_1}{\hbar} - \omega$$

$$\frac{d^2}{dt^2} C_1(t) = \frac{\gamma}{i\hbar}[-i\Delta\omega]\, e^{-i\Delta\omega t} C_2(t) + \frac{\gamma}{i\hbar} e^{-i\Delta\omega t} \frac{d}{dt} C_2(t)$$

$$= -\frac{\Delta\omega}{\hbar}\, i\hbar \frac{d}{dt} C_1(t) + \frac{\gamma}{i\hbar} e^{-i\Delta\omega t} \frac{1}{i\hbar} \gamma\, e^{i\Delta\omega t} C_1(t)$$

$$\Rightarrow \boxed{\frac{d^2}{dt^2} C_1(t) + i\,\Delta\omega \frac{d}{dt} C_1(t) + \frac{\gamma^2}{\hbar^2} C_1(t) = 0}$$

The characteristic Eq $\lambda^2 + i\Delta\omega\lambda + \frac{\gamma^2}{\hbar^2} = 0 \Rightarrow \lambda_{1,2} = \frac{-i\Delta\omega}{2} \pm i\sqrt{\left(\frac{\Delta\omega}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}$

define $\Omega = \left[\left(\gamma/\hbar\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2\right]^{1/2}$

$$\Rightarrow \begin{cases} C_1(t) = \left[A\, e^{i\Omega t} + B\, e^{-i\Omega t}\right] e^{-i\frac{\Delta\omega}{2}t} \\ C_2(t) = \frac{\hbar}{\gamma}\left[-(\Omega - \frac{\Delta\omega}{2}) A e^{i\Omega t} + (\Omega + \frac{\Delta\omega}{2}) B\, e^{-i\Omega t}\right] e^{i\frac{\Delta\omega}{2}t} \end{cases}$$

A, B to be determined

Consider the initial condition $C_1(0) = 1$, and $C_2(0) = 0$

we set
$$\begin{cases} A + B = 1 \\ -(\Omega - \frac{\Delta\omega}{2})A + (\Omega + \frac{\Delta\omega}{2})B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} + \frac{\Delta\omega}{4\Omega} \\ B = \frac{1}{2} - \frac{\Delta\omega}{4\Omega} \end{cases}$$

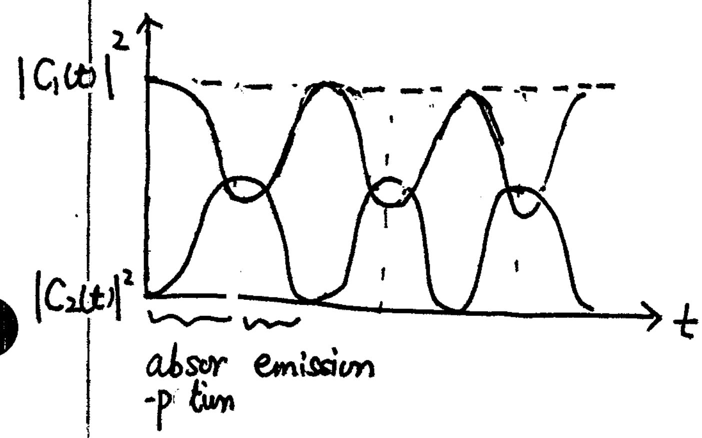
$$\Rightarrow C_2(t) = -\frac{i\hbar}{\gamma\Omega}\left[\Omega^2 - \left(\frac{\Delta\omega}{2}\right)^2\right] \sin\Omega t\, e^{-i\frac{\Delta\omega}{2}t} = \frac{-i\,\gamma/\hbar}{\Omega} \sin\Omega t\, e^{i\frac{\Delta\omega}{2}t}$$

$$|C_2(t)|^2 = \frac{(\gamma/\hbar)^2}{(\gamma/\hbar)^2 + \left(\frac{\Delta\omega}{2}\right)^2} \sin^2 \Omega t = \frac{(\gamma/\hbar)^2}{\Omega^2} \sin^2 \Omega t$$

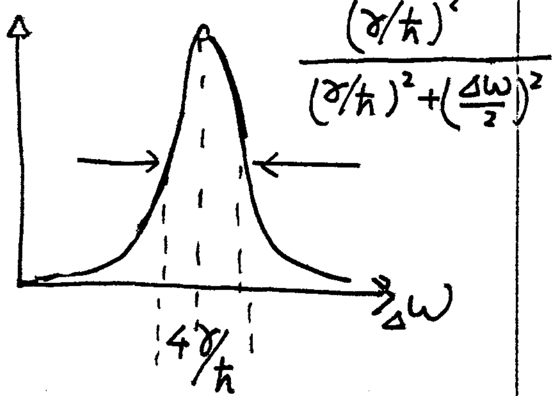
$$C_1(t) = \left[\cos\Omega t + \frac{i\Delta\omega}{2\Omega} \sin\Omega t\right] e^{-i\frac{\Delta\omega}{2}t}$$

$$|C_1(t)|^2 = \cos^2\Omega t + \left(\frac{\Delta\omega}{2\Omega}\right)^2 \sin^2\Omega t = 1 - \frac{\Omega^2 - \left(\frac{\Delta\omega}{2}\right)^2}{\Omega^2} \sin^2\Omega t$$

$$= 1 - |C_2(t)|^2$$



the magnitude of $C_2(t)$



$\Delta\omega = 0$ is resonance point!

§ Bloch sphere picture for 2 energy level system:

Now the wavefunction $\psi(t) = \begin{pmatrix} C_1(t)\, e^{-iE_1t/\hbar} \\ C_2(t)\, e^{-iE_2t/\hbar} \end{pmatrix}$

define

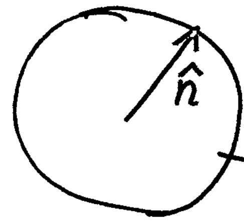
$$n_3 = |\psi_1|^2 - |\psi_2|^2$$

$$n_1 = \psi_1^*\psi_2 + \psi_2^*\psi_1$$

$$n_2 = -i\psi_1^*\psi_2 + i\psi_1^*\psi_2$$

or $\vec{n} = \psi_\alpha^* \vec{\sigma}_{\alpha\beta} \psi_\beta$

Ex check that $n_1^2 + n_2^2 + n_3^2 = 1$.



Bloch sphere

$$\begin{cases} n_3 = |C_1(t)|^2 - |C_2(t)|^2 \\ n_1 = 2\left[\mathrm{Re}\left[C_1^*(t)\, C_2(t)\right] \cos\frac{(E_2-E_1)t}{\hbar} + \mathrm{Im}\, C_1^*(t) C_2(t) \sin\frac{(E_2-E_1)t}{\hbar}\right] \\ n_2 = -2\mathrm{Re}\left[C_1^*(t)\, C_2(t)\right] \sin\frac{E_2-E_1}{\hbar}t + 2\,\mathrm{Im}\, C_1^*(t) C_2(t) \cos\frac{(E_2-E_1)t}{\hbar} \end{cases}$$

The n_1, n_2 components have a fast rotation due to the frequency $E_2 - E_1$, which is the Lamer frequency. Now we change to the corotating frame with the Lamer frequency, and only focus on the variation from $C_1(t)$ and $C_2(t)$. This is called - rotating wave frame. In this frame,

$$\psi'(t) = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} \longrightarrow \begin{cases} n_3' = n_3(t) \\ n_1' = 2\mathrm{Re}\, C_1^*(t) C_2(t) \\ n_2' = 2\,\mathrm{Im}\, C_1^*(t) C_2(t) \end{cases}$$

$$\vec{B} = B_0\hat{z} + B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$

$$\vec{\mu} = \frac{e}{m_e c}\vec{S}$$

$$H = -\vec{\mu}\cdot\vec{B} = H_0 + V(t)$$

$$H_0 = \hbar\omega_{21}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) \quad \text{where } \omega_{21} = \frac{|e|B_0}{m_e c}$$

$$V(t) = \gamma\cos\omega t\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right) + \gamma\sin\omega t\left(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|\right)$$

$$= \gamma e^{-i\omega t}|\uparrow\rangle\langle\downarrow| + \gamma e^{i\omega t}|\downarrow\rangle\langle\uparrow|, \quad \text{where } \gamma = \frac{-e\hbar B_1}{2m_e c}, \; e<0.$$

$|\uparrow\rangle = |2\rangle$

$|\downarrow\rangle = |1\rangle$

Then the problem is mapped back to previous one. ① If $B_1 = 0$, i.e. $\gamma = 0$, then $|C_\uparrow(t)|^2 = \text{const}$, $|C_\downarrow(t)|^2 = \text{const}$

spin only do precessions.

② If $B_1 \neq 0$, i.e. $\gamma \neq 0$, $|C_\uparrow(t)|^2$ and $|C_\downarrow(t)|^2$ varies according to

$$\Omega = \left[\left(\frac{\gamma}{\hbar}\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2\right]^{1/2}.$$

There're spin flop process!

• If for a linearly oscillating magnetic field, we can decompose it into a counter clockwise / clockwise part

$$2B_1\hat{x}\cos\omega t = B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t) + B_1(\hat{x}\cos\omega t - \hat{y}\sin\omega t)$$

$$= \gamma\left(e^{-i\omega t}|\uparrow\rangle\langle\downarrow| + e^{i\omega t}|\downarrow\rangle\langle\uparrow|\right) + \underbrace{\gamma\left(e^{i\omega t}|\uparrow\rangle\langle\downarrow| + e^{-i\omega t}|\downarrow\rangle\langle\uparrow|\right)}$$

$$i\hbar \frac{\partial}{\partial t} C_\downarrow(t)\, e^{-iE_1t/\hbar} |\downarrow\rangle + i\hbar \frac{\partial}{\partial t} C_\uparrow(t)\, e^{-iE_2t/\hbar} |\uparrow\rangle$$

$$= C_\downarrow(t)\, V(t)\, e^{-iE_1t/\hbar} |\downarrow\rangle + C_\uparrow(t)\, V(t)\, e^{-iE_2t/\hbar} |\uparrow\rangle$$

$$V(t) = \gamma\left(e^{-i\omega t} |\uparrow\rangle\langle\downarrow| + e^{i\omega t} |\downarrow\rangle\langle\uparrow|\right) + \gamma\left(e^{i\omega t} |\uparrow\rangle\langle\downarrow| + e^{-i\omega t} |\downarrow\rangle\langle\uparrow|\right)$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} C_\downarrow(t) e^{-iE_1t/\hbar} |\downarrow\rangle + i\hbar \frac{\partial}{\partial t} C_\uparrow(t)\, e^{-iE_2t/\hbar} |\uparrow\rangle$$

$$= C_\downarrow(t) \left[\gamma \left(e^{-i\omega t - iE_1t/\hbar} + e^{i\omega t - iE_1t/\hbar}\right) |\uparrow\rangle \right.$$

$$\left. + C_\uparrow(t) \left[\gamma \left(e^{i\omega t - iE_2t/\hbar} + e^{-i\omega t - iE_2t/\hbar}\right)\right] |\downarrow\rangle$$

$$\Rightarrow \begin{cases} i\hbar \frac{\partial}{\partial t} C_\downarrow(t) = \gamma\left[e^{i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t}\right] C_\uparrow(t) \\ i\hbar \frac{\partial}{\partial t} C_\uparrow(t) = \gamma\left[e^{-i(\omega-\omega_{21})t} + e^{i(\omega+\omega_{21})t}\right] C_\uparrow(t) \end{cases}$$

In the case of $\omega \to \omega_{21}$, we can neglect the parts of $e^{\pm i(\omega+\omega_{21})t}$ which are fast oscillation, which can be droped. In other words, B_0 already selects the handness.