Prob :1

$$d^j_{m'm}(\beta) = \langle jm' | \, e^{-iJ_y\beta} \, | jm \rangle$$

$$|jm\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \, |0\rangle$$

$$\frac{e^{-iJ_y\beta}\,(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}\,e^{iJ_y\beta}}{\sqrt{(j+m)!\,(j-m)!}} = \frac{(a_1^{\dagger\prime})^{j+m}(a_2^{\dagger\prime})^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \, |0\rangle$$

$$(a_1^\dagger)' = e^{-iJ_y\beta}\, a_1^\dagger \, e^{iJ_y\beta} \qquad \Rightarrow \qquad \frac{d a^\dagger_{1,2}(\beta)}{d\beta} = e^{-iJ_y\beta}\,[J_y\, a^\dagger_{1,2}]\,e^{iJ_y\beta}$$

$$(a_2^\dagger)' = e^{-iJ_y\beta}\, a_2^\dagger \, e^{iJ_y\beta}$$

$$J_y = \frac{1}{2i}\,(a_1^\dagger a_2 - a_2^\dagger a_1)$$

$$[J_y\, a_1^\dagger] = -\frac{1}{2i}\, a_2^\dagger$$

$$[J_y\, a_2^\dagger] = \frac{1}{2i}\, a_1^\dagger$$

$$\Rightarrow \left\{ \begin{aligned} \frac{d a_1^{\dagger\prime}(\beta)}{d\beta} &= \frac{1}{2}\, a_2^{\dagger\prime}(\beta) \\ \frac{d a_2^{\dagger\prime}(\beta)}{d\beta} &= -\frac{1}{2}\, a_1^{\dagger\prime}(\beta) \end{aligned} \right. \qquad \Rightarrow \qquad \boxed{\begin{aligned} a_1^{\dagger\prime}(\beta) &= a_1^\dagger \cos\frac{\beta}{2} + a_2^\dagger \sin\frac{\beta}{2} \\ a_2^{\dagger\prime}(\beta) &= -a_1^\dagger \sin\frac{\beta}{2} + a_2^\dagger \cos\frac{\beta}{2} \end{aligned}}$$

for $j = 3/2$

$$e^{-iJ_y\beta}\,|m\rangle = \frac{1}{\sqrt{(m+3/2)!\,(3/2-m)!}}\left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger \sin\frac{\beta}{2}\right)^{\frac{3}{2}+m}$$

$$\left(-a_1^\dagger \sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)^{\frac{3}{2}-m} |0\rangle$$

$$e^{-iJ_y\beta}\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \frac{1}{\sqrt{3!}}\Big[(a_1^\dagger)^3\cos^3\frac{\beta}{2} + 3(a_1^\dagger)^2 a_2^\dagger\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} + 3a_1^\dagger(a_2^\dagger)^2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}$$
$$+ (a_2^\dagger)^3\sin^3\frac{\beta}{2}\Big]|0\rangle$$

$$= \cos^3\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle + \underbrace{\sqrt{\frac{2!}{3!}}\,3}_{\sqrt{3}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle + \underbrace{\sqrt{\frac{2!}{3!}}\,3}_{\sqrt{3}}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\left|\tfrac{3}{2}\,{-\tfrac{1}{2}}\right\rangle + \sin^3\frac{\beta}{2}\left|\tfrac{3}{2}\,{-\tfrac{3}{2}}\right\rangle$$

$$e^{-iJ_y\beta}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{2!}}\left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right)^2\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\Big[(a_1^\dagger)^2\cos^2\frac{\beta}{2} + 2a_1^\dagger a_2^\dagger\cos\frac{\beta}{2}\sin\frac{\beta}{2} + (a_2^\dagger)^2\sin^2\frac{\beta}{2}\Big]\Big[-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\Big]|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\Big[-\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}(a_1^\dagger)^3 + \Big[\cos^3\frac{\beta}{2} - 2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\Big)(a_1^\dagger)^2 a_2$$
$$+ \Big[2\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} - \sin^3\frac{\beta}{2}\Big]a_1^\dagger(a_2^\dagger)^2 + (a_2^\dagger)^3\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\Big]|0\rangle$$

$$= -\sqrt{\frac{3!}{2!}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle + \cos\frac{\beta}{2}\left(\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2}\right)\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$$
$$+ \sin\frac{\beta}{2}\Big[2\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\Big]\left|\tfrac{3}{2}\,{-\tfrac{1}{2}}\right\rangle + \sqrt{\frac{3!}{2!}}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\left|\tfrac{3}{2}\,{-\tfrac{3}{2}}\right\rangle$$

$$e^{-iJ_y\beta}\left|\tfrac{3}{2}\,{-\tfrac{1}{2}}\right\rangle = \frac{1}{\sqrt{2!}}\Big[a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\Big]\Big[-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\Big]^2|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\Big[a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\Big]\Big[(a_1^\dagger)^2\sin^2\frac{\beta}{2} - 2a_1^\dagger a_2^\dagger\sin\frac{\beta}{2}\cos\frac{\beta}{2} + (a_2^\dagger)^2\cos^2\frac{\beta}{2}\Big]|0\rangle$$

$$= \sqrt{\frac{3!}{2!}}\Big[\frac{1}{\sqrt{3!}}(a_1^\dagger)^3\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\Big] + \frac{1}{\sqrt{2}}(a_1^\dagger)^2(a_2^\dagger)\Big[-2\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} + \sin^3\frac{\beta}{2}\Big]$$
$$+ \frac{1}{\sqrt{2}}a_1^\dagger(a_2^\dagger)^2\Big[\cos^3\frac{\beta}{2} - 2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\Big] + \sqrt{\frac{3!}{2!}}\frac{1}{\sqrt{3!}}(a_2^\dagger)^3\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}|0\rangle$$

$$= \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle + \sin\frac{\beta}{2}\left[\sin^2\frac{\beta}{2} - 2\cos^2\frac{\beta}{2}\right]\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$$

$$+ \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2}\right]\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle$$

$$e^{-iJ_y\beta}\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle = \frac{1}{\sqrt{3!}}\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)^3|0\rangle$$

$$= \frac{1}{\sqrt{3!}}\left(-(a_1^\dagger)^3\sin^3\frac{\beta}{2} + 3(a_1^\dagger)^2(a_2^\dagger)\sin^2\frac{\beta}{2}\cos\frac{\beta}{2} - 3a_1^\dagger(a_2^\dagger)^2\sin\frac{\beta}{2}\cos^2\frac{\beta}{2} + (a_2^\dagger)^3\cos^3\frac{\beta}{2}\right)|0\rangle$$

$$= (-)^3\sin^3\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle + \sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle - \sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle + \cos^3\frac{\beta}{2}\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle$$

Collect every item together $\qquad d^{3/2}_{m'm}(\beta) =$

$$\begin{array}{cccc}
\multicolumn{4}{c}{\left(\begin{array}{cccc}
\cos^3\frac{\beta}{2} & -\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & -\sin^3\frac{\beta}{2} \\
\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2}\right] & -\sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \\
\sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] & \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2}\right] & -\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \\
\sin^3\frac{\beta}{2} & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & \sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \cos^3\frac{\beta}{2}
\end{array}\right)} \\
\tfrac{3}{2} & \tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{3}{2}
\end{array}$$

$$\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2} = \frac{3\cos\beta - 1}{2}$$

$$2\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2} = \frac{1 + 3\cos\beta}{2}$$

Week 8 Discussion

Mid-term Exam

Problem 2 Perturbation Theory

Planar rotor in the external quadrupolar potential

Solution:

1) $\Psi_m^0 = \frac{1}{\sqrt{2\pi}} e^{im\phi}$. $E_m^0 = \frac{\hbar^2}{2I} \cdot m^2$.

$E_m^0 = E_{-m}^0$. hence except $m=0$, there is a double degeneracy for $\pm m$.

Reflection symmetry (sending ϕ to $-\phi$) or time reversal symmetry protects this double degeneracy.

2) Before applying H', the system has both $SO(2)$ rotation and reflection symmetries. Hence the unitary symmetry group of H_0 is $O(2)$.

After H' is introduced, $SO(2)$ rotation symmetry is broken, and the system has only a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, i.e. $\{\mathbb{1}, R_x\} \times \{\mathbb{1}, R_y\}$, where $\begin{cases} R_x : \phi \mapsto -\phi \\ R_y : \phi \mapsto \pi - \phi \end{cases}$.

We do not expect energy level degeneracy now, since the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is Abelian.

When $V_0 \ll \frac{\hbar^2}{2I}$, we can treat H' as a perturbation to H_0.

3) For later convenience, we calculate the following matrix element:

$$\langle \Psi_n^{(0)} | H' | \Psi_m^{(0)} \rangle$$
$$= \int_0^{2\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{-in\phi} \left(-\frac{V_0}{2}\right)\left(e^{i2\phi} + e^{-i2\phi}\right) \frac{1}{\sqrt{2\pi}} e^{im\phi}$$
$$= -\frac{V_0}{2}\left(\delta_{n,m+2} + \delta_{n,m-2}\right)$$

where $m, n \in \mathbb{Z}$.

For ground state $\Psi_{m=0}^0$, we use non-degenerate perturbation theory.

Applying eqn. 5.1.42 and 5.1.44 in Sakurai's book 2nd edition, we get

1st order wavefunction correction:

$$\Psi_{m=0}^{(1)} = \sum_{n\neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \Psi_n^{(0)} \langle \Psi_n^{(0)} | H' | \Psi_0^{(0)} \rangle$$

$$= \sum_{n\neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \Psi_n^{(0)} \left(-\frac{V_0}{2}\right)\left(\delta_{n,2} + \delta_{n,-2}\right)$$

$$\left(E_0^{(0)} - E_n^{(0)} = -n^2 \hbar^2/2I \right)$$

$$= \frac{1}{4} \frac{V_0}{\hbar^2/I} \left(\Psi_2^{(0)} + \Psi_{-2}^{(0)}\right)$$

2nd order energy correction: (1st order energy correction vanishes)

$$E_{m=0}^{(2)} = \sum_{n\neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \left|\langle \Psi_n^{(0)} | H' | \Psi_0^{(0)} \rangle\right|^2$$

$$= \sum_{n\neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \frac{1}{4} V_0^2 \left(\delta_{n,2} + \delta_{n,-2}\right)^2$$

$$= \frac{\frac{1}{4} V_0^2}{-4\hbar^2/2I} \sum_{n\neq 0} \left(\delta_{n,2} + \delta_{n,-2}\right)^2$$

$$= -\frac{1}{4} \frac{V_0^2}{\hbar^2/I}$$

$\square$

Hence in summary:

$$\Psi_{m=0} = \Psi^{(0)}_{m=0} + \frac{1}{4}\frac{V_0}{\hbar^2/I}\left(\Psi^{(0)}_{2} + \Psi^{(0)}_{-2}\right) + O\left(\left(\frac{V_0}{\hbar^2/I}\right)^2\right)$$

$$E_{m=0} = -\frac{1}{4}\frac{V_0^2}{\hbar^2/I} + O\left(\left(\frac{V_0}{\hbar^2/I}\right)^3\right)$$

It is obvious that the new ground state is parity even (at least up to first order of $V_0/\hbar^2/I$ according to our calculation). But actually it is of even parity to all orders. This is because the ground state is non-degenerate and hence ~~has a~~ is parity eigenstate. The original ground state $\Psi^{(0)}_{m=0}$ has even parity, and thus a continuous analysis shows that new ground state is also of even parity provided that V_0 is small enough.

4) For $m=\pm1$, we should use ~~non-~~ degenerate perturbation theory.

First we determine the 0th order wavefunction. For this we calculate the projection of H' onto this 2-dim degenerate space.

$$\langle \Psi^{(0)}_{1} | H' | \Psi^{(0)}_{1}\rangle = \langle \Psi^{(0)}_{-1} | H' | \Psi^{(0)}_{-1}\rangle = 0$$

$$\langle \Psi^{0}_{1} | H' | \Psi^{(0)}_{-1}\rangle = \langle \Psi^{(0)}_{-1} | H' | \Psi^{(0)}_{1}\rangle = -\frac{V_0}{2}$$

Hence $P_0 H' P_0 = \begin{bmatrix} 0 & -\frac{V_0}{2} \\ -\frac{V_0}{2} & 0 \end{bmatrix}$, where $P_0 = |\Psi^{(0)}_{1}\rangle\langle\Psi^{(0)}_{1}| + |\Psi^{(0)}_{-1}\rangle\langle\Psi^{(0)}_{-1}|$.

Diagonalizing this 2x2 matrix we get

1st order energy correction	0th order wavefunction
$E^{(1)}_{1,+} = \frac{V_0}{2}$	$\frac{1}{\sqrt{2}}\left(\Psi^{(0)}_{1} - \Psi^{(0)}_{-1}\right)$
$E^{(1)}_{1,-} = -\frac{V_0}{2}$	$\frac{1}{\sqrt{2}}\left(\Psi^{(0)}_{1} + \Psi^{(0)}_{-1}\right)$

Using eqn. 5.2.15 in Sakurai's book 2nd edition, we get the 2nd order energy correction as follows

$$E^{(2)}_{1\pm} = \sum_{n\neq\pm1} \frac{|\langle \Psi^{(0)}_{n} | H' | \Psi^{(0)}_{1,\pm}\rangle|^2}{E^{(0)}_{1,\pm} - E^{(0)}_{n}}$$

in which

$$E^{(0)}_{1,\pm} = E^{(0)}_{\pm1} = \frac{\hbar^2}{2I}$$

$$\langle \Psi^{(0)}_{n} | H' | \Psi^{(0)}_{1,\pm}\rangle = \langle \Psi^{(0)}_{n} | H' | \frac{1}{\sqrt{2}}\left(|\Psi^{(0)}_{1}\rangle \mp |\Psi^{(0)}_{-1}\rangle\right)$$

$$= \left(-\frac{V_0}{2}\right)\frac{1}{\sqrt{2}}\left((\delta_{n,3} + \delta_{n,-1}) \mp (\delta_{n,1} + \delta_{n,-3})\right)$$

hence

$$E^{(2)}_{1,\pm} = \frac{V_0^2}{8}\sum_{n\neq\pm1} \frac{\left((\delta_{n,3}+\delta_{n,-1}) \mp (\delta_{n,1}+\delta_{n,-3})\right)^2}{E^{(0)}_{1,\pm} - E^{(0)}_{n}}$$

$$= \frac{V_0^2}{8}\frac{1}{(1^2-3^2)\hbar^2/2I}(1+1)$$

$$= -\frac{1}{16}\frac{V_0^2}{\hbar^2/8I}$$

For 1^{st} order wavefunction, we use eqn. 5.2.6 and 5.2.14 in Sakurai's book 2^{nd} edition. Notice that there are two parts of this 1^{st} order correction: ~~One~~ one lying in span$\{\Psi_1^{(0)}, \Psi_{-1}^{(0)}\}$, the other outside this degenerate space.

Then

$$P_0 |\Psi^{(1)}_{1,\pm}\rangle = \frac{|\Psi^{(0)}_{1,\mp}\rangle}{E^{(1)}_{1,\mp} - E^{(1)}_{1,\pm}} \langle \Psi^{(0)}_{1,\mp} | H' P_1 \frac{1}{E^{(0)}_{1,\pm} - H_0} P_1 H' | \Psi^{(0)}_{1,\pm}\rangle$$

(where $P_1 = \mathbb{1} - P_0$)

$$= \frac{\Psi^{(0)}_{1,\mp}}{\mp V_0} \sum_{n\neq\pm1} \frac{1}{\sqrt{2}} \left(\langle\Psi_1^{(0)}| \pm \langle \Psi_{-1}^{(0)}|\right) H' |\Psi_n^{(0)}\rangle \cdot \frac{1}{\frac{\hbar^2}{2I}(1-n^2)} \cdot \langle \Psi_n^{(0)} | H' | \frac{1}{\sqrt{2}} \left(|\Psi_1^{(0)}\rangle \mp |\Psi_{-1}^{(0)}\rangle \right)$$

$$= \frac{1}{\mp V_0} \Psi^{(0)}_{1,\mp} \ \frac{1}{2} \cdot \frac{1}{4} V_0^2 \cdot \frac{1}{\hbar^2/2I} \cdot \sum_{n\neq\pm1} \frac{1}{1-n^2} \left((\delta_{n,3}+\delta_{n,-1}) \pm (\delta_{n,1}+\delta_{n,-3})\right) \cdot \left((\delta_{n,3}+\delta_{n,-1}) \mp (\delta_{n,1}+\delta_{n,-3})\right)$$

$$= \mp \frac{V_0}{\hbar^2/I} \frac{1}{4} \Psi^{(0)}_{1,\mp} \ \frac{1}{1-3^2} \left(1 + (\pm1)\cdot(\mp1)\right)$$

$$= 0$$

$$P_1 |\Psi^{(1)}_{1,\pm}\rangle = \sum_{n\neq\pm1} \frac{|\Psi_n^{(0)}\rangle \langle \Psi_n^{(0)} | H' | \Psi^{(0)}_{1,\pm}\rangle}{E^{(0)}_{1,\pm} - E_n^{(0)}}$$

$$= \sum_{n\neq\pm1} \frac{|\Psi_n^{(0)}\rangle \cdot \langle \Psi_n^{(0)} | H' | \frac{1}{\sqrt{2}} \left(|\Psi_1^{(0)}\rangle \mp |\Psi_{-1}^{(0)}\rangle\right)}{\hbar^2/2I \cdot (1-n^2)}$$

$$= \sum_{n\neq\pm1} |\Psi_n^{(0)}\rangle \cdot \left(-\frac{V_0}{2}\right) \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\hbar^2/2I} \cdot \frac{1}{1-n^2} \left((\delta_{n,3}+\delta_{n,-1}) \mp (\delta_{n,1}+\delta_{n,-3})\right)$$

$$= -\frac{V_0}{\hbar^2/I} \frac{1}{\sqrt{2}} \frac{1}{1-3^2} \left(\Psi_3^{(0)} \mp \Psi_{-3}^{(0)}\right)$$

$$= \frac{1}{8\sqrt{2}} \frac{V_0}{\hbar^2/I} \left(\Psi_3^{(0)} \mp \Psi_{-3}^{(0)}\right)$$

In summary.

$$E_{1,\pm} = \frac{\hbar^2}{2I} \pm \frac{V_0}{2} - \frac{1}{16} \frac{V_0^2}{\hbar^2/I} + O\left(\left(\frac{V_0}{\hbar^2/I}\right)^3\right)$$

$$\Psi_{1,\pm} = \frac{1}{\sqrt{2}}\left(\Psi_1^{(0)} \mp \Psi_{-1}^{(0)}\right) + \frac{1}{8\sqrt{2}} \frac{V_0}{\hbar^2/I} \left(\Psi_3^{(0)} \mp \Psi_{-3}^{(0)}\right) + O\left(\left(\frac{V_0}{\hbar^2/I}\right)^2\right).$$

Again at least to 1^{st} order of $\frac{V_0}{\hbar^2/I}$, $\Psi_{1,\pm}$ are of parity odd and even respectively. A similar continuous analysis shows that this ~~holds to~~ holds to all orders.

Prob 3

a) Use $\vec{\mathcal{E}}$ for the electric field polarization. The transition rate $\propto |\langle f|\vec{\mathcal{E}}\cdot\vec{r}\,|i\rangle|^2$. For the 2p states, considering SO coupling we have $2P_{3/2}$ and $2P_{1/2}$ set. Their radial wave function are the same $R_{2p}(r)$. The angular parts are SO coupled spherical harmonic

$$\psi_{2p,\frac{3}{2}\frac{3}{2}} = R_{2p}(r)\begin{pmatrix} Y_{11}(\theta,\varphi) \\ 0 \end{pmatrix}$$

$$\psi_{2p\,\frac{3}{2}\frac{1}{2}} = R_{2p}(r)\begin{pmatrix} \sqrt{\frac{2}{3}}\,Y_{10}(\theta,\varphi) \\ \sqrt{\frac{1}{3}}\,Y_{11}(\theta,\varphi) \end{pmatrix} \qquad \psi_{2p\frac{1}{2}\frac{1}{2}} = R_{2p}(r)\begin{pmatrix} -\sqrt{\frac{1}{3}}\,Y_{10}(\theta,\varphi) \\ \sqrt{\frac{2}{3}}\,Y_{11}(\theta,\varphi) \end{pmatrix}$$

$$\psi_{2p\,\frac{3}{2}-\frac{1}{2}} = R_{2p}(r)\begin{pmatrix} \sqrt{1/3}\,Y_{1-1}(\theta,\varphi) \\ \sqrt{2/3}\,Y_{10}(\theta,\varphi) \end{pmatrix} \qquad \psi_{2p\frac{1}{2}-\frac{1}{2}} = R_{2p}(r)\begin{pmatrix} -\sqrt{2/3}\,Y_{1-1}(\theta,\varphi) \\ \sqrt{\frac{1}{3}}\,Y_{10}(\theta,\varphi) \end{pmatrix}$$

$$\psi_{2p\,\frac{3}{2}-\frac{3}{2}} = R_{2p}(r)\begin{pmatrix} 0 \\ Y_{1-1}(\theta,\varphi) \end{pmatrix}$$

The initial state $\psi_{1s,\frac{1}{2}\frac{1}{2}} = R_{1s}(r)\begin{pmatrix} Y_{00} \\ 0 \end{pmatrix}$.

Consider $\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$

$$= r\sqrt{\frac{4\pi}{3}}\left[\frac{1}{\sqrt{2}}(Y_{1-1} - Y_{11})\hat{e}_1 + \frac{i}{\sqrt{2}}(Y_{1-1} + Y_{11})\hat{e}_2 + Y_{10}\,\hat{e}_3\right]$$

$$\Rightarrow \langle Y^{2P}_{\frac{3}{2}\frac{3}{2}}(\theta,\varphi)|\,\vec{r}\,|\,Y^{1S}_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi)\rangle = r\sqrt{\frac{4\pi}{3}}\Big\{\langle Y_{11}|\,\underbrace{\tfrac{1}{\sqrt{2}}(-)Y_{11}}_{\hat{e}_1}|Y_{00}\rangle$$

$$\langle Y_{11} | Y_{11} | Y_{00} \rangle = \int d\Omega \, Y_{11}^* Y_{11} \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \langle \mathcal{Y}^{2p}_{3/2\,3/2}(\theta,\varphi) | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2}(\theta,\varphi) \rangle = r\sqrt{\frac{1}{6}} \, (-\hat{e}_1 + i\hat{e}_2)$$

$$\langle \mathcal{Y}^{2p}_{3/2\,1/2} | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle = \left[\sqrt{\frac{2}{3}} \langle Y_{10} | \vec{r} | Y_{00} \rangle \right.$$

$$= \underbrace{\sqrt{\frac{4\pi}{3}}}_{r} \left[\sqrt{\frac{2}{3}} \langle Y_{10} | \underbrace{Y_{10}}_{\hat{e}_3} | Y_{00} \rangle \right.$$

$$= r\frac{\sqrt{2}}{3} \hat{e}_3$$

$$\langle \mathcal{Y}^{2p}_{3/2\,-1/2} | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle = +\sqrt{\frac{1}{3}} \langle Y_{1-1} | \vec{r} | Y_{00} \rangle = r\sqrt{\frac{4\pi}{3}} (+)\sqrt{\frac{1}{3}} \langle Y_{1-1} | \frac{1}{\sqrt{2}} Y_{1-1} \hat{e}_1 + \frac{i}{\sqrt{2}} Y_{1-1} \hat{e}_2 | Y_{00} \rangle$$

$$= \frac{r}{3} \frac{1}{\sqrt{2}} [\hat{e}_1 + i\hat{e}_2] = \frac{\sqrt{2}}{6} r [\hat{e}_1 + i\hat{e}_2]$$

$$\langle \mathcal{Y}^{2p}_{3/2\,-3/2} | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle = 0 ;$$

$$\langle \mathcal{Y}^{2p}_{1/2\,1/2} | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle = -\sqrt{\frac{1}{3}} \langle Y_{10} | \vec{r} | Y_{00} \rangle = -\sqrt{\frac{1}{3}} \, r\sqrt{\frac{4\pi}{3}} \langle Y_{10} | Y_{10} | Y_{00} \rangle \hat{e}_3$$

$$= -\frac{r}{3} \hat{e}_3$$

$$\langle \mathcal{Y}^{2p}_{1/2\,-1/2} | \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle = -\sqrt{\frac{2}{3}} \langle Y_{1-1} | \vec{r} | Y_{00} \rangle = -\sqrt{\frac{2}{3}} \, r\sqrt{\frac{4\pi}{3}} \langle Y_{1-1} | \frac{1}{\sqrt{2}} (\hat{e}_1 + i\hat{e}_2) Y_{1-1} | Y_{00} \rangle$$

$$= -\frac{r}{3} (\hat{e}_1 + i\hat{e}_2)$$

$$\mathcal{E} = \mathcal{E}_x \hat{e}_1 + \mathcal{E}_y \hat{e}_2 + \mathcal{E}_z \hat{e}_3$$

$$\Rightarrow \left| \langle \mathcal{Y}^{2p}_{3/2\,3/2} | \vec{\mathcal{E}} \cdot \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle \right|^2 = \frac{r^2}{6} (\mathcal{E}_x^2 + \mathcal{E}_y^2)$$

$$\left| \langle \mathcal{Y}^{2p}_{3/2\,1/2} | \vec{\mathcal{E}} \cdot \vec{r} | \mathcal{Y}^{1s}_{1/2\,1/2} \rangle \right|^2 = \frac{2r^2}{9} \mathcal{E}_z^2$$

$$|\langle y^{2p}_{3/2\,-1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{r^2}{18}(\varepsilon_x^2+\varepsilon_y^2)$$

$$|\langle y^{2p}_{3/2\,-3/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = 0$$

$$|\langle y^{2p}_{1/2\,1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{r^2}{9}\varepsilon_z^2$$

$$|\langle y^{2p}_{1/2\,-1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{r^2}{9}(\varepsilon_x^2+\varepsilon_y^2)$$

$\Rightarrow$ total transition rate from $\psi_{1s,\uparrow}$ to $2P_{3/2}$ is

$$W_{3/2} \propto \sum_{m=-3/2}^{3/2} |\langle y^{2p}_{3/2,m} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{2}{9}r^2\varepsilon^2$$

$$W_{1/2} \propto \sum_{m=-1/2}^{1/2} |\langle y^{2p}_{1/2\,m} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{1}{9}r^2\varepsilon^2$$

$\Rightarrow$ the ratio is independent of the polarization of $\vec{\varepsilon}$,

$W_{3/2} : W_{1/2} = 2:1$.

b) If , if the final states $\psi^{2p}_{3/2\,m}$ for $m = 3/2, 1/2, -1/2, -3/2$

$\vec{k} \parallel \hat{x}$ $\Rightarrow \varepsilon_x = 0$, $\langle\varepsilon_y^2\rangle = \langle\varepsilon_z^2\rangle = \frac{\varepsilon^2}{2}$. $\Rightarrow$

$$\left.\begin{aligned}
&|\langle y^{2p}_{3/2\,3/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{r^2}{6}\varepsilon^2\cdot\frac{1}{2}\\
&|\langle y^{2p}_{3/2\,1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{2}{9}r^2\varepsilon^2\cdot\frac{1}{2}\\
&|\langle y^{2p}_{3/2\,-1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = \frac{r^2}{18}\frac{\varepsilon^2}{2}\\
&|\langle y^{2p}_{3/2\,-1/2} | \vec{\varepsilon}\cdot\vec{r} | y^{1s}_{1/2\,1/2}\rangle|^2 = 0
\end{aligned}\right\}$$

$\Rightarrow$ branch ratio

$\frac{1}{12} : \frac{1}{9} : \frac{1}{36} : 0$

$= 3:4:1:0$