PHYS 212A: Homework 2

October 29, 2013

1.18

a

Sakurai gives the proof within the text in 1.4.56.

b

See Sakurai's derivation of the general uncertainty principle in section 1.4. Since expectation values are commutitive numbers (or by elementary calculation) we have

$$[A, B] = [\Delta A, \Delta B]. \tag{1}$$

Thus, we can evaluate

$$\langle\alpha| [\Delta A, \Delta B] |\alpha\rangle = \lambda^* \langle\alpha| \Delta B \Delta B |\alpha\rangle - \lambda \langle\alpha| \Delta B \Delta B |\alpha\rangle \tag{2}$$

and

$$\langle\alpha| \{\Delta A, \Delta B\} |\alpha\rangle = \lambda^* \langle\alpha| \Delta B \Delta B |\alpha\rangle + \lambda \langle\alpha| \Delta B \Delta B |\alpha\rangle \tag{3}$$

For λ purely imaginary, the two terms will cancel and the expectation of the anticommutator becomes zero. Now look at line 1.4.63 in Sakurai's derivation. Since the anticommutator term vanishes, the equality is exact.

c

We'll expand the expectation values over the complete set $\langle x''|$, with the normalization condition $\langle x'|x''\rangle = \delta(x' - x'')$. Now consider

$$\langle x'| \Delta p |\alpha\rangle = \int dx'' \delta(x' - x'')(-i\hbar \frac{\partial}{\partial x''} - < p >) \langle x''|\alpha\rangle \tag{4}$$

Taking advantage of the explicit form of $\langle x''|\alpha\rangle$ given in the problem, we can evaluate the derivative term explicitly to find

$$\begin{gathered} \langle x'| \Delta p |\alpha\rangle \\ = \int dx'' \delta(x' - x'')(-i\hbar(\tfrac{ip}{\hbar} - \tfrac{2(x'' - \langle x\rangle)}{4d^2} - < p >) \langle x''|\alpha\rangle \\ = \tfrac{i\hbar}{2d^2} \int dx'' \delta(x' - x'')(x'' - \langle x\rangle) \langle x''|\alpha\rangle \\ = \tfrac{i\hbar}{2d^2} \langle x'|\Delta x|\alpha\rangle \end{gathered}$$

1.21

Recall that for the infinite square well, the wave functions are of the form $\sqrt{(2/a)}\sin(n\pi x/a)$. To evaluate the uncertainty product we'll take advantage of the fact that $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, and $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$. Since we know the explicit form of the wavefunctions we can evaluate integrals to find expectation values. The results of this procedure are

$$\langle x^2\rangle = a^2[1/3 - 1/2n^2\pi^2] \tag{5}$$

$$\langle x\rangle = a/2 \tag{6}$$

$$\langle p^2\rangle = \hbar^2/a2(n\pi)^2 \tag{7}$$

$$\langle p\rangle = 0 \tag{8}$$

Combining these results, we can calculate the uncertainty product with the result

$$\langle(\Delta x)^2\rangle\,\langle(\Delta p)^2\rangle = \frac{\hbar^2}{2}[(n\pi)^2/6 - 1] \tag{9}$$

1.28

a

$$[x, F(p_x)]_{CL} = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F}{\partial x} = \frac{\partial F}{\partial p_x} \tag{10}$$

b

Using $i\hbar[x, F(p_x)]_{CL} = [x, F(p_x)]_{QM}$ we have

$$[x, \exp(ip_x a/\hbar)]_{QM} = i\hbar\frac{\partial}{\partial p_x}\exp(ip_x a/\hbar) = -a\exp(ip_x a/\hbar) \tag{11}$$

c

From part b we have

$$[x, \exp(ip_x a/\hbar)]\,|x'\rangle = -a\exp(ip_x a/\hbar)\,|x'\rangle \tag{12}$$

which can be rearranged to yield

$$x[\exp(ip_x a/\hbar)\,|x'\rangle] = (x' - a)[\exp(ip_x a/\hbar)\,|x'\rangle]. \tag{13}$$

This last equality implies that $\exp(ip_x a/\hbar)\,|x'\rangle$ is an eigenstate of the coordinate operator x with the corresponding eigen value $(x' - a)$.

1.30

The solution to this problem is nearly identical to the solution of 1.28 b and c.

1.33

a

We'll start by expanind $\langle p'|x|p''\rangle$ over a complete set of x'.

$$\begin{gathered}\langle p'|x|p''\rangle \\ = \int \langle p'|x|x'\rangle \langle x'|p''\rangle dx' \\ = \int x' \langle p'|x'\rangle \langle x'|p''\rangle dx' \\ = \tfrac{1}{2\pi\hbar} \int dx' x' e^{-ix'(p'-p'')/\hbar}\end{gathered}$$

Now we recognize

$$\frac{1}{2\pi\hbar} \int dx' e^{-ix'(p'-p'')/\hbar} = \delta(p'-p'') \tag{14}$$

which implies

$$\langle p'|x|p''\rangle = i\hbar \frac{\partial}{\partial p'} \delta(p'-p'') \tag{15}$$

We now expand over a complete set of momentum states and can write

$$\begin{gathered}\langle p'|x|\alpha\rangle \\ = \int dp'' \langle p'|x|p''\rangle \langle p''|\alpha\rangle = \int dp'' i\hbar \frac{\partial}{\partial p'} \delta(p'-p'') \langle p''|\alpha\rangle \\ = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle\end{gathered}$$

For part ii we have

$$\begin{gathered}\langle \beta|x|\alpha\rangle \\ = \int dp' \langle \beta|p'\rangle \langle p'|x|\alpha\rangle \\ = \int dp' \langle \beta|p'\rangle i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \\ = \int dp' \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha^*(p')\end{gathered}$$

b

This portion of the problem is nearly identical to 1.28, only with the roles of position and momentum switched. Evaluation of the commutator $[x, \exp(ix\Xi/\hbar)]$ yields the result,

$$p[\exp(ix\Xi/\hbar) |p'\rangle] = (p' + \Xi)[\exp(ix\Xi/\hbar) |p'\rangle] \tag{16}$$

which shows that the operator $\exp(ix\Xi/\hbar)$ is the momentum translation operator and that x is the generator of momentum translation.