Lecture 6: Pictures

Read Sakurai and Napolitano Chapter 2.1 and 2.2

Both operators and state vectors are unobservable, only the inner products are related to observable quantities. Under the requirement of keeping the inner products invariant, we can use different pictures to formulate the time-evolution in quantum mechanics. They are related to similar transformations.

1 Schrödinger picture

The time evolution is expressed as the evolution of state vectors. The canonical coordinates, momenta, spin do not change with time.

$$|\Psi^S(t)\rangle = T(t,t_0)|\Psi^s(t_0)\rangle. \tag{1}$$

Assume that F^s is an operator for the observable F in the Schrödinger picture, we have its expectation value

$$\overline{F^s} = \langle\Psi^s(t)|F^s|\Psi^s(t)\rangle. \tag{2}$$

If H^s does not depend on t, then $T(t,0) = \exp\{-\frac{i}{\hbar}Ht\}$. If H^s explicitly depend on t, the expression of T is not so simple. According to $\frac{\partial}{\partial t}T(t,0) = -\frac{i}{\hbar}H^s(t)T(t,0)$, we have

$$T(t,0) = 1 + \frac{-i}{\hbar}\int_0^t dt_1 H^s(t_1)T(t_1,0), \tag{3}$$

and by iteration,

$$\begin{aligned}
T(t,0) &= 1 + \frac{-i}{\hbar}\int_0^t dt_1 H^s(t_1) + (\frac{-i}{\hbar})^2\int_0^t dt_1\int_0^{t_1} dt_2 H^s(t_1)H^s(t_2) \\
&+ ... + (\frac{-i}{\hbar})^n\int_0^t dt_1 ... \int_0^{t_{n-1}} dt_n H^s(t_1)H^s(t_2)...H^s(t_n) + ... \\
&= \sum_{n=0}^{+\infty}\frac{1}{n!}(\frac{-i}{\hbar})^n\int_0^t dt_1 ... \int_0^t dt_n \mathcal{T} H^s(t_1)H^s(t_2)...H^s(t_n) \\
&= \mathcal{T}\exp\{\frac{-i}{\hbar}\int_0^t H(t^{'})dt^{'}\}.
\end{aligned} \tag{4}$$

$\mathcal{T}$ is the time-ordered operator. $\mathcal{T}$ is defined as

$$\mathcal{T} H(t_1)H(t_2)...H(t_n) = \sum_p \theta(t_{p_1} > t_{p_2} > ... > t_{p_n})H(t_{p_1})H(t_{p_2})...H(t_{p_n}), \tag{5}$$

where p is a permutation of $1,2,...,n$, and θ-function equals to 1 if the condition $t_{p_1} > t_{p_2} > ... > t_{p_n}$ is satisfied and otherwise 0.

2 Heisenberg picture

We can also fix state vector stationary with time, say, set the state vector in Heisenberg picture as the that of the Schrödinger one at $t = 0$, and let operator to evolve with time:

$$\begin{aligned} |\Psi^H\rangle &= |\Psi^s(0)\rangle, \\ \hat{F}^H(t) &= T^t(t,0)\hat{F}^s T(t,0), \end{aligned} \tag{6}$$

such that

$$\langle\Psi^H|F^H|\Psi^H\rangle = \langle\Psi^s|F^s|\Psi^s\rangle. \tag{7}$$

Actually, F^s can also explicitly depend on time as $F^s(t)$.

Now let us derive the equation of motion of operators. From the Schrödinger equation in which that H^s may explicitly depend on time, $i\hbar\frac{\partial}{\partial t}|\Psi^s(t)\rangle = H^s(t)|\Psi^s(t)\rangle$, we have $i\hbar\frac{\partial}{\partial t}T(t,0)|\Psi^s(0)\rangle = H^s(t)T(t,0)|\Psi^s(0)\rangle$, thus we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}T(t,0) &= H^s(t)T(t,0) \\ -i\hbar\frac{\partial}{\partial t}T^\dagger(t,0) &= T^\dagger(t,0)H^s(t). \end{aligned} \tag{8}$$

Then we have $\frac{d}{dt}F^H(t) = \frac{\partial}{\partial t}T^t(t,0)F^s(t)T(t,0) + T^t(t,0)\frac{\partial}{\partial t}F^s(t)T(t,0) + T^t(t,0)F^s(t)\frac{\partial}{\partial t}T(t,0)$, and then

$$\frac{d}{dt}F^H(t) = \frac{1}{i\hbar}[F^H(t), H^H(t)] + T^\dagger(t,0)\frac{\partial F^s(t)}{\partial t}T(t,0), \tag{9}$$

where $H^H(t) = T^\dagger(t,0)H^s(t)T(t,0)$. In particular, for the canonical coordinate and momentum, we have

$$\begin{aligned} \frac{d}{dt}q_j^H(t) &= \frac{1}{i\hbar}[q_j^H(t), H^H(t)], \\ \frac{d}{dt}p_j^H(t) &= \frac{1}{i\hbar}[p_j^H(t), H^H(t)]. \end{aligned} \tag{10}$$

Example 1) Consider the Hamiltonian of an harmonic oscillator in the Schrödinger picture

$$H^S = \frac{p^{S,2}}{2m} + \frac{1}{2}m\omega^2 x^{S,2}. \tag{11}$$

Then the operators in the Heisenberg picture x^H and p^H can be solved in the following way.

$$\begin{aligned}
\frac{d}{dt}x^H(t) &= \frac{1}{i\hbar}[x^H(t),H^H] = \frac{1}{i\hbar}e^{iHt}[x,H^S]e^{-iHt}, \\
\frac{d}{dt}p^H(t) &= \frac{1}{i\hbar}[p^H(t),H^H] = \frac{1}{i\hbar}e^{iHt}[p,H^S]e^{-iHt}.
\end{aligned} \tag{12}$$

Using the fact that

$$[x^S,H^S] = \frac{1}{2m}[x,p^{S,2}] = \frac{i\hbar}{m}p^S \tag{13}$$

and

$$[p^S,H^s] = \frac{1}{2}m\omega^2[p^S,x^2] = -i\hbar m\omega^2 x^S, \tag{14}$$

we arrive at

$$\frac{d}{dt}x^H(t) = \frac{p^H(t)}{m}, \quad \frac{d}{dt}p^H(t) = -m\omega^2 x^H(t), \tag{15}$$

the solution will be

$$\begin{aligned}
x^H(t) &= x^S\cos\omega t + \frac{p^S}{m\omega}\sin\omega t, \\
p^H(t) &= -m\omega x^S\sin\omega t + p^S\cos\omega t.
\end{aligned} \tag{16}$$

Exercise 1 Please prove Eq. 16 for harmonic oscillators by directly using

$$x^H(t) = e^{\frac{i}{\hbar}H^S t}x^S e^{-\frac{i}{\hbar}H^S t}, \quad p^H(t) = e^{\frac{i}{\hbar}H^S t}p^S e^{-\frac{i}{\hbar}H^S t}. \tag{17}$$

Hint: You may use the Baker-Hausdorff lemma in page 95 in Sakurai and Napolitano's book. This formula is proved in my notes in lecture 7.

3 Interaction picture

We decompose the Hamiltonian H^s of the Schrödinger picture into the free part H_0 and the perturbative part V as

$$H^s = H_0 + V, \tag{18}$$

which H_0 is independent of time; V may depend on time. We define that the state vector evolves with time as

$$|\Psi^I(t)\rangle = e^{iH_0t/\hbar}|\Psi^S(t)\rangle = e^{iH_0t/\hbar}T(t,0)|\Psi^S(0)\rangle, \tag{19}$$

and correspondingly the operator

$$F^I(t) = e^{iH_0 t/\hbar} F^s e^{-iH_0 t/\hbar}. \tag{20}$$

In such a convention, we keep the inner product invariant

$$\langle \Psi_A^I(t) | F^I(t) | \Psi_B^I(t) \rangle = \langle \Psi_A^S(t) | F^S(t) | \Psi_B^S(t) \rangle \tag{21}$$

Now let us derive equation of motion, we have

$$\frac{d}{dt} F^I(t) = \frac{1}{i\hbar} [F^I(t), H_0] + e^{iH_0 t/\hbar} \frac{\partial F^s(t)}{\partial t} e^{-iH_0 t/\hbar}. \tag{22}$$

For state vector, we have

$$\begin{aligned} \frac{\partial}{\partial t} |\Psi^I(t)\rangle &= \frac{i}{\hbar} H_0 e^{iH_0 t/\hbar} |\Psi^s(t)\rangle + e^{iH_0 t/\hbar} \frac{1}{i\hbar} H^S |\Psi^S(t)\rangle \\ &= e^{iH_0 t/\hbar} \frac{i}{\hbar} (H_0 - H^s) e^{-iH_0 t/\hbar} |\Psi^I(t)\rangle = \frac{1}{i\hbar} V^I(t) |\Psi^I(t)\rangle. \end{aligned} \tag{23}$$

From Eq. 23, we can derive the time-evolution operator $U(t, t_0)$ in the interaction picture as

$$\begin{aligned} |\Psi^I(t)\rangle &= U(t, t_0) |\Psi^I(t_0)\rangle, \\ U(t, t_0) &= \mathcal{T} \exp\{-\frac{i}{\hbar} \int_{t_0}^{t} dt' V^I(t')\}. \end{aligned} \tag{24}$$

Exercise 2

1. Prove Eq. 22 of $\frac{d}{dt} F^I(t)$.
2. Prove the above expression of $U(t, t_0)$.

Example Let us consider a perturbed Harmonic potential

$$H = H_0 + V(x), \tag{25}$$

where $H_0 = p^2/2m + \frac{1}{2} m\omega^2 x^2$ and $V(x)$ is a small perturbation. In the interaction picture, we have

$$\begin{aligned} x^I(t) = e^{\frac{i}{\hbar} H_0 t} x e^{-\frac{i}{\hbar} H_0 t} &= x \cos \omega t + \frac{p}{m\omega} \sin \omega t, \\ p^I(t) = e^{\frac{i}{\hbar} H_0 t} p e^{-\frac{i}{\hbar} H_0 t} &= -m\omega x \sin \omega t + p \cos \omega t, \end{aligned} \tag{26}$$

and thus

$$V^I(t) = V(x^I(t)). \tag{27}$$

The time-evolution operator for the state vectors are

$$U(t,t_0) = \mathcal{T}\exp\{-\frac{i}{\hbar}\int_{t_0}^{t} dt V(x^I(t))\}. \tag{28}$$