

12:

Lect Motion in the magnetic field - Landau levels

$$H_{2D}^{LL} = \frac{(p - \frac{e}{c}A)^2}{2m}$$

$$\vec{A} = \frac{B}{2} \hat{z} \times \vec{r} = \frac{B}{2} (-y, x)$$

$$\text{cyclotron radius } l_B = \sqrt{\frac{\hbar c}{|eB|}} = \frac{257 \text{ \AA}}{\sqrt{B/\text{Tesla}}}$$

$$H_{2D}^{LL} = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 r^2 - \frac{eB}{2mc} \hat{z} \cdot (\vec{r} \times \vec{p}), \quad \omega_0 = \frac{|eB|}{2mc}$$

$$H_{2D} = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 r^2 \mp \omega_0 L_z$$

where '+' apply for $\vec{B} \parallel \hat{z}$ respectively.

2D Landau level Hamiltonian is nothing but 2D Harmonic oscillator plus orbital Zeeman term. The $\omega_0 L_z$ term commutes with $\frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 r^2$, thus the 2D Landau level wavefunctions are nothing but 2D harmonic oscillator wavefunctions. But we know 2D harmonic oscillators do not have very exciting properties, but the orbital Zeeman term reorganizes the spectra, and leads to non-trivial topology.

Let us first look at the spectra and wavefunctions of 2D harmonic oscillator

$$H_{\text{harm}} = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 r^2$$

$$\left[-\frac{\hbar^2}{2m} \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{2} m \omega_0^2 r^2 \right] \psi = E \psi$$

Separation of variables $\psi(r, \phi) = R(r) e^{im'\phi}$

define length unit $l = \sqrt{\frac{\hbar}{m\omega_0}}$

$$\Rightarrow \left[l^2 \frac{d^2}{dr^2} + \frac{l^2}{r} \frac{d}{dr} - \frac{m'^2 l^2}{r^2} + \frac{2E}{\hbar\omega_0} - \frac{r^2}{l^2} \right] R(r) = 0$$

as $r \rightarrow 0, \Rightarrow \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m'^2}{r^2} \right] R(r) = 0$

$$\Rightarrow R(r) \sim r^{|m'|}$$

as $r \rightarrow \infty \Rightarrow \left[l^2 \frac{d^2}{dr^2} - \frac{r^2}{l^2} \right] R(r) = 0 \Rightarrow R(r) \sim e^{-\frac{r^2}{2l^2}}$

\Rightarrow we can try $R(r) = r^{|m'|} e^{-\frac{r^2}{2l^2}} u(r) \Rightarrow u(r)$ satisfies

$$l^2 \frac{d^2 u}{dr^2} + \left[\frac{2|m'|+1}{r/l} - \frac{2r}{l} \right] l \frac{du}{dr} + \left[\frac{2E}{\hbar\omega_0} - 2(|m'|+1) \right] u = 0$$

define $\xi = \frac{r^2}{l^2} \Rightarrow$

$$\xi \frac{d^2 u}{d\xi^2} + [|m'|+1 - \xi] \frac{du}{d\xi} - \left[\frac{|m'|+1}{2} - \frac{E}{2\hbar\omega_0} \right] u = 0$$

\rightarrow Confluent hypergeometric equation, $\Rightarrow \frac{|m'|+1}{2} - \frac{E}{2\hbar\omega_0} = -n_r$

$$E_{n_r, m'} = (2n_r + |m'| + 1) \hbar\omega_0$$

$$\psi_{n_r, m'}(r, \phi) \sim e^{im'\phi} r^{|m'|} e^{-\frac{r^2}{2l^2}} F(-n_r, |m'|+1, \frac{r^2}{l^2})$$

$$F(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} z + \frac{\alpha(\alpha+1)}{2! \gamma(\gamma+1)} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3! \gamma(\gamma+1)(\gamma+2)} z^3 + \dots$$

$$F(-n_r, |m'|+1, \frac{r^2}{l^2}) = 1 + \left(\frac{-n_r}{|m'|+1}\right) \frac{r^2}{l^2} + \frac{(-n_r)(-n_r+1)}{2! (|m'|+1)(|m'|+2)} \left(\frac{r}{l}\right)^4$$

$$+ \frac{(-n_r)(-n_r+1) \dots (-1)}{n_r! (|m'|+1)(|m'|+2) \dots (|m'|+n_r)} \left(\frac{r}{l}\right)^{2n_r} \leftarrow \text{Polynomial}$$

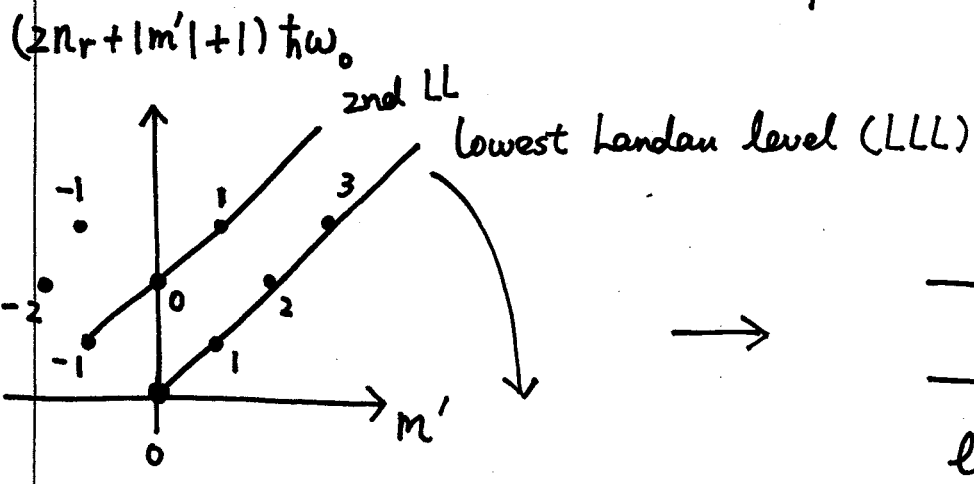
only has finite terms.

$$E_{\text{Landau}} = E_{n_r, m'} - m' \hbar \omega_0 = (2n_r + |m'| + 1 - m') \hbar \omega_0$$

$$= \hbar \omega_c [n_r + \frac{1}{2}] \text{ if } m' > 0$$

$$\left\{ \hbar \omega_c [n_r + |m'| + \frac{1}{2}] \text{ if } m' < 0 \right.$$

$$\omega_c = 2\omega_0 = \frac{|eB|}{m c}$$



$$\left. \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\} \omega_c = \frac{|eB|}{m c}$$

$$l_B = \sqrt{\frac{\hbar c}{eB}} = \frac{1}{\sqrt{2}} l$$

What's special of the lowest Landau level?

$$\psi_{n_r=0, m'}^{\text{LLL}} = \frac{1}{\sqrt{2\pi} l_B^2 2^{m'} m'!} \left(\frac{z}{l_B}\right)^{m'} e^{-\frac{|z|^2}{4 l_B^2}}, \text{ where } z = x + iy$$

infinite degeneracy with respect to $m' = 0, 1, 2, 3, \dots$

* Classic radius (LLL states)

$$\rho = |\psi|^2 \propto r^{2m} e^{-\frac{r^2}{2l_B^2}} \Rightarrow \frac{\partial \rho}{\partial (r^2)} = \left[m(r^2)^{m-1} + (r^2)^m \left(-\frac{1}{2l_B^2}\right) \right] e^{-\frac{r^2}{2l_B^2}} = 0$$

$$\Rightarrow r_c^2 = 2m l_B^2$$

between the m -th, and the $m+1$ -th orbits, there's one state

$$\Rightarrow \text{average density } \rho \sim \frac{1}{\pi (r_c^2(m+1) - r_c^2(m))} = \frac{1}{2\pi l_B^2} \quad \left\{ \begin{array}{l} \text{Consider we fill the} \\ \text{system with fermions} \end{array} \right.$$

Actually, this is exact result if LLL is fully filled!

$$\begin{aligned} \rho(r) &= \sum_{m=0}^{\infty} |\psi_{LLL,m}(r)|^2 = \frac{1}{2\pi l_B^2} \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{|z|^2}{2l_B^2}\right)^m \right] e^{-\frac{|z|^2}{2l_B^2}} \\ &= \frac{1}{2\pi l_B^2}, \text{ which is a const} \end{aligned}$$

* Magnetic translation

the mechanical momenta are $\vec{P} = \vec{p} - \frac{e}{c} \vec{A}$

$$\Rightarrow P_x = -i\hbar \partial_x + \frac{eB}{2c} y = -i\hbar \partial_x + \frac{\hbar y}{2l_B^2}$$

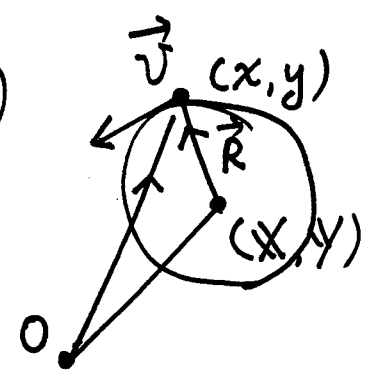
$$P_y = -i\hbar \partial_y - \frac{eB}{2c} x = -i\hbar \partial_y - \frac{\hbar x}{2l_B^2}$$

define guiding center

$$X = x + \frac{l_B^2}{\hbar} p_y = \frac{l_B^2}{\hbar} (-i\hbar \partial_y + \frac{\hbar}{2l_B^2} x)$$

$$Y = y - \frac{c}{eB} p_x = -\frac{l_B^2}{\hbar} (-i\hbar \partial_x - \frac{\hbar}{2l_B^2} y)$$

$$H_{2D}^{LL} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}$$



Check that the guiding centers are conserved quantities

$$[X, H^{LL}] = [Y, H^{LL}] = 0.$$

$$\begin{aligned} \vec{R} &= \frac{c}{eB} (p_y, -p_x) \\ &= \frac{l_B^2}{\hbar} (p_y, -p_x) \end{aligned}$$

Thus we can put the guiding center onto exponential

$$T_x(d_x) = e^{\frac{+i y d_x}{eB}} = e^{-i(-i\partial_x - \frac{y}{2l_B^2}) \cdot d_x} = e^{-\partial_x \partial_x + i \frac{y d_x}{2l_B^2}}$$

$$T_y(d_y) = e^{-\partial_y \partial_y - i \frac{x d_y}{2l_B^2}}$$

(d_x, d_y are translation distance)

$$[T_x(d_x), H_{2D}^{LL}] = [T_y(d_y), H_{2D}^{LL}] = 0$$

generally speaking, for a translation along a vector $\vec{\delta}$

$$T[\vec{\delta}] = e^{-\vec{\delta} \cdot \vec{\nabla} + \frac{i}{2l_B^2} \hat{z} \cdot (\vec{\delta} \times \vec{r})}$$
$$[T[\vec{\delta}], H_{2D}^{LL}] = 0$$

But $T_x[\delta_x]$ and $T_y[\delta_y]$ donot commute

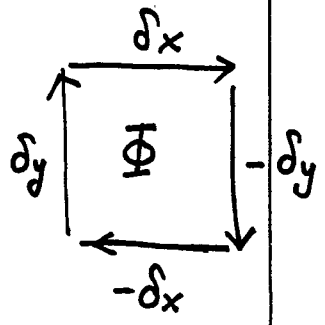
$$T_x[\delta_x] T_y[\delta_y] = e^{\delta_x[-\partial_x + \frac{iy}{2\ell_B^2}]} e^{\delta_y[-\partial_y - \frac{i\partial_x}{2\ell_B^2}]}$$

according to the fomula $e^A e^B = e^B e^A e^{[A,B]}$

if $[A,B], A = [A,B], B = 0$. We can check that $[-\partial_x + \frac{iy}{2\ell_B^2}, -\partial_y - \frac{i\partial_x}{2\ell_B^2}] = \frac{i}{\ell_B^2}$

$$\Rightarrow T_x[\delta_x] T_y[\delta_y] = T_y[\delta_y] T_x[\delta_x] e^{\frac{i \delta_x \delta_y}{\ell_B^2}}$$

$$\Rightarrow T_x^{-1}[\delta_x] T_y^{-1}[\delta_y] T_x[\delta_x] T_y[\delta_y] = e^{i \delta_x \delta_y / \ell_B^2} = e^{i 2\pi \Phi / \Phi_0}$$

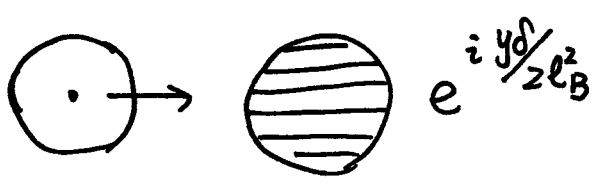


Let us try to translate the Gaussian pocket $\psi_{LLL, m=0} \propto e^{-\frac{|z|^2}{4\ell_B^2}}$

$$T_x[\delta] e^{-\frac{|z|^2}{4\ell_B^2}} = e^{-\delta \partial_x + \frac{i}{2\ell_B^2} \delta y} e^{-\frac{|z|^2}{4\ell_B^2}} = e^{-\frac{(x-\delta)^2 + y^2}{4\ell_B^2}} e^{\frac{i y \delta}{2\ell_B^2}} = e^{-\frac{|z|^2}{4\ell_B^2}} \cdot e^{\frac{\delta}{2\ell_B^2} (x+iy)}$$

$f(z)$: only dependents on z .

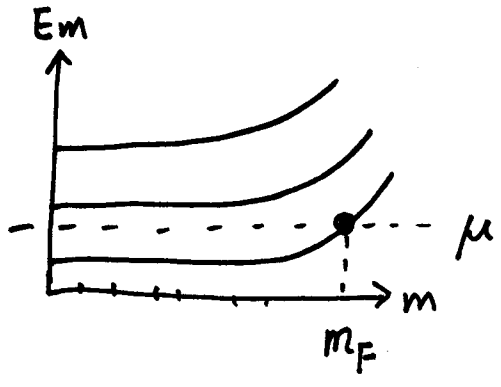
after translation, the state remains in the LLL.



edge - spectra

Now let us consider put the system in a disc with radius R . What will be the spectra in this case with the open boundary condition?

as m is small, $r_{c,m} = \sqrt{2m} l_B \ll R$, thus for these states, they don't see the open boundary, thus their spectra remain flat.



However, for those states with large values of m , its classic radius \rightarrow the wall, then their energy ~~is~~ are pushed up. As m goes very large, their wavefunctions are thrown to the wall, thus they are called edge states. The criterion is $\sqrt{2m} l_B \gg R$. for these m 's, they are edge states. In this case, its hamiltonian becomes a circular rotor,

$$H_{2D} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 - \omega_0 L_z$$

$$m\hbar = Rk$$

linear momentum

$$H_{1D} \xrightarrow{\text{edge}} \frac{m^2 \hbar^2}{2MR^2} + \text{const} - m\hbar\omega_0 \quad (m > 0)$$

we will fill with fermions, at chemical potential μ . If it cuts the spectra at $m \simeq m_F$, we can linearize the spectra as

$$H_{1D} \simeq \frac{v}{R} (m - m_F) \hbar \rightsquigarrow v(k - k_F) \leftarrow \text{chiral fermi liquid.}$$

if we consider edge as flat

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