Lect 15: Superconductivity — functional field method

$$H = \int dr\, \psi^{+}_{\alpha}(r) \left[\frac{1}{2m}(-i\nabla - eA)^2 + ie\phi - \mu\right] \psi_{\alpha}(r) - g\int dr\, \psi^{+}_{\uparrow}(r)\psi^{+}_{\downarrow}(r)\, \psi_{\downarrow}(r)\, \psi_{\uparrow}(r)$$

this ϕ is $i \otimes$ conventional scaler potential

$$\rightarrow S[\bar\psi, \psi] = \int_0^\beta d\tau \int d\vec r \left\{ \left[\bar\psi_\alpha \left(\partial_\tau + ie\phi + \frac{1}{2m}(-i\nabla - eA)^2 - \mu\right] \psi_\alpha(r) - g\, \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow \right\}$$

gauge invariance
$$\psi_\alpha \to e^{i\theta}\psi_\alpha, \qquad \vec A \to \vec A + e^{-1}\nabla\theta = \vec A'$$
$$\bar\psi_\alpha \to e^{-i\theta}\bar\psi_\alpha \qquad \phi \to \phi - e^{-1}\partial_\tau\theta = \phi'$$

minimal coupling
$$(-i\nabla - e\vec A')\psi' = e^{i\theta}(-i\nabla - eA)\psi$$
$$(\partial_\tau + ie\phi')\psi' = e^{i\theta}(\partial_\tau + ie\phi)\psi$$

§ Hubbard - Stratonovich transformation:

$$\exp\left[g\int d\tau\int dr\, \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow\right] = \int D\bar\Delta D\Delta\, \exp\left[-\int d\tau \int dr \left(\frac{|\Delta|^2}{g} - (\bar\Delta \psi_\downarrow\psi_\uparrow + \Delta\bar\psi_\uparrow\bar\psi_\downarrow)\right)\right]$$

define Nambu spinor

$$\bar\psi = (\bar\psi_\uparrow, \psi_\downarrow), \qquad \psi = \begin{pmatrix} \psi_\uparrow \\ \bar\psi_\downarrow \end{pmatrix}$$

$$\mathcal{Z} = \int D\bar\psi D\psi \int D\bar\Delta D\Delta\, \exp\left\{-\int d\tau\, dr \left[\frac{1}{g}|\Delta|^2 - \bar\psi \mathcal{G}^{-1}\psi\right]\right\},$$

where $\mathcal{G}^{-1} = \begin{bmatrix} -\partial_\tau - ie\phi - \frac{1}{2m}(-i\nabla - eA)^2 + \mu, & \Delta \\ \bar{\Delta}, & -\partial_\tau + ie\phi + \frac{1}{2m}(i\nabla - eA)^2 - \mu \end{bmatrix}$

$G_p^{-1} = -\partial_\tau - H, \quad G_h^{-1} = -\partial_\tau + H^T,$ where $\hat{x}^T = \hat{x}$, $\hat{p}^T = -\hat{p}$.

$$\Rightarrow Z = \int D\bar{\Delta}\, D\Delta \; \exp\left[-\frac{1}{g}\int d\tau\, dr\, |\Delta|^2 + \ln \det -\mathcal{G}^{-1}\right]$$

$$= \int D\bar{\Delta}\, D\Delta \; \exp\left\{-\frac{1}{g}\int d\tau\, dr\, |\Delta|^2 + \mathrm{tr}\, \ln[-\mathcal{G}^{-1}]\right\}$$

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1}(\Delta_0) + \delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-, \quad \text{where } \Delta = \Delta_0 + \delta\Delta$$

$$= \mathcal{G}_0^{-1}\left[1 + \mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-)\right]$$

$$\Rightarrow \mathrm{tr}[\ln(-\mathcal{G}^{-1})] = \mathrm{tr}[\ln(-\mathcal{G}_0^{-1}(1 + \mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-)))]$$

$$= \mathrm{tr}\{\ln(-\mathcal{G}_0^{-1})\} + \mathrm{tr}[\ln(1 + \mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-))]$$

$$= \mathrm{tr}\{\ln(-\mathcal{G}_0^{-1})\} + \mathrm{tr}[\mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-)]$$

$$- \frac{1}{2}\mathrm{tr}[\mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-)\, \mathcal{G}_0(\delta\Delta\, \tau^+ + \delta\bar{\Delta}\, \tau^-)]$$

$+ \cdots\cdots$

$$|\Delta + \delta\Delta|^2 = |\Delta|^2 + \bar{\Delta}_0\, \delta\Delta + \Delta_0\, \delta\bar{\Delta} + \delta\bar{\Delta}\, \delta\Delta$$

$\Rightarrow$ search for the saddle point to the linear order

$$\left\{-\frac{1}{g}\bar{\Delta}_0 + \mathrm{tr}_{x,\tau}\left[\mathcal{G}_0(\Delta_0)(x\tau, x\tau)\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]\right\}\delta\Delta = 0$$

$$-\frac{1}{g}\bar{\Delta}_0 + \frac{1}{V\beta}\sum_{k,i\omega_n}\begin{bmatrix} i\omega_n - \xi_k, & \Delta_0 \\ \bar{\Delta}_0, & i\omega_n + \xi_k\end{bmatrix}^{-1}_{21} = 0$$

$$\Rightarrow -\frac{1}{g}\bar{\Delta}_0 + \frac{1}{V}\sum_k \frac{1}{\beta}\sum_{i\omega_n}\frac{1}{(i\omega_n)^2 - E_k^2}\begin{bmatrix} i\omega_n + \xi_k, & -\Delta \\ -\bar{\Delta}_0, & i\omega_n - \xi_k\end{bmatrix}_{21} = 0$$

$$\Rightarrow -\frac{1}{g} + \int\frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{i\omega_n}\frac{-1}{(i\omega_n)^2 - E_k^2} = 0$$

$$\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{(i\omega_n)^2 - E_k^2} = +\frac{1}{\beta}\frac{1}{2E_k}\sum_{i\omega_n}\left(\frac{1}{i\omega_n - E_k} - \frac{1}{i\omega_n + E_k}\right)$$

$$= \frac{1}{2E_k}\left[n_F(E_k) - n_F(-E_k)\right] = -\frac{1}{2E_k}\tanh\frac{\beta}{2}E_k$$

$\Rightarrow$ gap Equation: $$\boxed{\frac{1}{g} = \int\frac{d^3k}{(2\pi)^3}\frac{\tanh\frac{\beta}{2}E_k}{2E_k}}.$$

§ Expansion close to T_c — Ginzburg - Landau free energy

Let us set the free $\mathcal{G}_0$ as $\Delta = 0$, and expand the free energy around it. $\mathcal{G}_0$ is propagator of the free system.

$$\Rightarrow \operatorname{tr}\ln\left[-\mathcal{G}_0^{-1}(1+\mathcal{G}_0\hat{\Delta})\right] = \underbrace{\operatorname{tr}\ln -\mathcal{G}_0^{-1}}_{\text{const}} + \underbrace{\operatorname{tr}\ln(1+\hat{\mathcal{G}}_0\hat{\Delta})}_{-\sum_{n=0}^{\infty}\frac{1}{2n}\operatorname{tr}\left[(\hat{\mathcal{G}}_0\hat{\Delta})^{2n}\right]} \leftarrow \boxed{\text{even power}}$$

where $\mathcal{G}_0(k, i\omega_n) = \begin{bmatrix} i\omega_n - \xi_k & 0 \\ 0 & i\omega_n + \xi_k \end{bmatrix}$, $\hat{\Delta} = \begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}$.

$$\Rightarrow -\frac{1}{2}\operatorname{tr}\left[(\hat{\mathcal{G}}_0\hat{\Delta})^2\right] = -\frac{1}{2}\operatorname{tr}\left[\mathcal{G}_0\begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}\mathcal{G}\begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}\right]$$

$$= -\operatorname{tr}\left[\mathcal{G}_{0,11}\,\Delta\,\mathcal{G}_{0,22}\,\bar{\Delta}\right] = -\sum_q \sum_p \left[\mathcal{G}_{0,11}(p)\,\Delta(q)\,\mathcal{G}_{0,22}(p-q)\,\bar{\Delta}(q)\right]$$

$$\langle p|\mathcal{G}_{0,11}|p\rangle\langle p|\Delta|p-q\rangle\langle p-q|\mathcal{G}_{0,22}|p-q\rangle\langle p-q|\bar{\Delta}|p\rangle$$

$$\Rightarrow -\frac{1}{2}\operatorname{tr}\left[(\hat{\mathcal{G}}_0\hat{\Delta})^2\right] = -\sum_q \bar{\Delta}(q)\Delta(q)\sum_{\vec{p}}\frac{1}{\beta}\sum_{ip_n}\mathcal{G}_{0,11}(p)\,\mathcal{G}_{0,22}(p-q)$$

$$\Rightarrow S^2[\Delta,\bar{\Delta}] = \sum_q \Gamma_q^{-1}|\Delta(q)|^2$$

$$= \sum_q\left(\frac{1}{g} + \frac{1}{V\beta}\sum_{\vec{p}}\sum_{ip_n}\frac{1}{ip_n - \xi_p}\,\frac{1}{ip_n - iq_n + \xi_{p-q}}\right)$$

$$= \sum_q\left(\frac{1}{g} - \frac{1}{V\beta}\sum_{\vec{p}}\sum_{ip_n}\frac{1}{ip_n - \xi_{\vec{p}}}\,\frac{1}{iq_n - ip_n - \xi_{-\vec{p}+\vec{q}}}\right)$$

$$-\frac{1}{\beta}\sum_{ip_n}\frac{1}{ip_n - \xi_{\vec{p}}}\,\frac{1}{iq_n - ip_n - \xi_{-\vec{p}+\vec{q}}} = \frac{1 - n_f(\xi_p) - n_f(\xi_{-p+q})}{iq_n - \xi_p - \xi_{-p+q}}$$

$$\Gamma^{-1}(q=0) = \frac{1}{g} + \frac{1}{V}\sum_{\vec p} \frac{(1-2n_f(\xi_p))}{-2\xi_p} = \frac{1}{g} - \int\frac{d^3k}{(2\pi)^3}\frac{\tanh\frac{\beta}{2}\xi_p}{2\xi_p}$$

$\Gamma^{-1}(q=0)=0$ means the instability, i.e. T_c is determined by

$$\frac{1}{g} = N(0)\int_{-\hbar\omega_D}^{+\hbar\omega_D} d\xi \, \frac{\tanh\frac{\beta}{2}\xi_p}{2\xi_p}.$$

Expand $\Gamma^{-1}(q=0,T)$ around T_c $\Rightarrow$

$$\tanh\frac{\beta}{2}\xi_p = \tanh\frac{\beta_c}{2}\xi_p + \frac{-\frac{\beta_c}{2}\xi_p}{ch^2\frac{\beta_c}{2}\xi_p}\frac{\Delta T}{T_c}$$

$$\Rightarrow \Gamma^{-1}(q=0,T) = \left[\int\frac{d^3k}{(2\pi)^3}\frac{\frac{\beta_c}{2}}{ch^2\frac{\beta_c}{2}\xi_p}\right]\frac{\Delta T}{T_c} = N(0)\left[\int d\xi\,\frac{\frac{\beta_c}{2}}{ch^2\frac{\beta_c\xi}{2}}\right]\frac{\Delta T}{T_c}$$

$$= N(0)\left[\int_{-\infty}^{+\infty} dx\,\frac{1}{ch^2 x}\right]\frac{\Delta T}{T_c} = N(0)\frac{\Delta T}{T_c}.$$

Similarly for higher order: set $(q, q_n)=0$ $\Rightarrow$

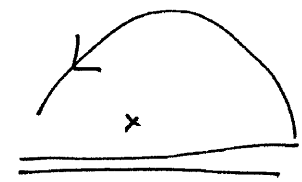
$$S^{(2n)} = \frac{1}{2n}\,tr\{[\mathcal{G}_0\hat\Delta]^{2n}\} = \frac{1}{2n}\,tr\left\{\left[\begin{matrix} G_p & \\ & -G_{-p}\end{matrix}\right]\left[\begin{matrix} 0 & \Delta \\ \bar\Delta & 0\end{matrix}\right]\right\}^{2n}$$

$$= \frac{1}{2n}\,tr\left\{\left[\begin{matrix} 0, & G_p\Delta \\ -G_{-p}\bar\Delta, & 0\end{matrix}\right]^2\right\}^n = \frac{(-)^n}{2n}\,tr\left[\begin{matrix} G_p\Delta\, G_{-p}\bar\Delta, & 0 \\ 0 & G_{-p}\bar\Delta\, G_p\Delta\end{matrix}\right]^n$$

$$= \frac{(-)^n}{n}\sum_p \frac{1}{\beta}\sum_{ip_n}\left[\frac{1}{ip_n-\xi_p}\,\frac{1}{-ip_n-\xi_{-p}}\right]^n |\Delta|^{2n}$$

$$= \frac{(-)^n}{n} \frac{N(0)}{\beta} \sum_{ip_n} \int_{-\omega_D}^{\omega_D} d\xi \left(\frac{1}{p_n^2 + \xi^2} \right)^n |\Delta|^{2n}$$

$$\int_{-\infty}^{+\infty} d\xi \left(\frac{1}{p_n^2 + \xi^2} \right)^n = |p_n|^{-(2n-1)} \int_{-\infty}^{+\infty} dx \left(\frac{1}{x^2+1} \right)^n$$



$$\int_{-\infty}^{+\infty} dx \left(\frac{1}{x^2+1} \right)^n = \frac{2\pi i}{(n-1)!} \left[\frac{d^{n-1}}{dx^{n-1}} \left(\frac{1}{x+i} \right)^n \right] \Bigg|_{x=i}$$

$$= \frac{2\pi}{(n-1)!} \frac{(2n-1)!}{(n-1)!} \left(\frac{1}{2i} \right)^{2n-1} (-)^{n-1} \cdot i = \frac{2\pi \, (2n-1)!}{[(n-1)!]^2 \, 2^{2n-1}}$$

$$\Rightarrow S^{(2n)} = (-)^n N(0) T \left(\frac{|\Delta|}{T} \right)^{2n} \cdot \text{const of } (n).$$

$$\Rightarrow \underline{S_{GL}[\Delta, \bar{\Delta}] = \int d^d r \;\; r|\Delta|^2 + \beta |\Delta|^4 \;\; + \text{gradient term}}.$$

§6 Action of the Goldstone mode (in the ordered state $\Delta \neq 0$)

$$\mathcal{G}^{-1} = \begin{pmatrix} -\partial_\tau - i\phi - \frac{1}{2m}(-i\nabla - A)^2 + \mu, & \Delta_0 e^{i2\theta} \\ \Delta_0 e^{-i2\theta}, & -\partial_\tau + i\phi + \frac{1}{2m}(i\nabla - A)^2 - \mu \end{pmatrix}$$

where $e\phi \to \phi$, $eA \to A$, "e" has been absorbed. Further, we can absorb "θ" by the transform $\hat{U} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, then

$$\mathcal{G}^{-1} \to U \mathcal{G}^{-1} U^{\dagger} = \begin{pmatrix} -\partial_\tau - i\tilde{\phi} - \frac{1}{2m}(-i\nabla - \tilde{A})^2 + \mu, & \Delta_0 \\ \Delta_0, & -\partial_\tau + i\tilde{\phi} + \frac{1}{2m}(i\nabla - \tilde{A})^2 - \mu \end{pmatrix}$$

where $\tilde{A} = A - \nabla\theta$, $\tilde{\phi} = \phi + \partial_\tau\theta$.

Now $$\mathcal{G}^{-1} = \underbrace{\begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu, & \Delta_0 \\ \Delta_0, & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}}_{\mathcal{G}_0^{-1}} \underbrace{- i\tilde{\phi}\tau_3 - \frac{i}{2m}[\nabla, \tilde{A}]_+}_{-\hat{X}_1} \underbrace{- \frac{\tilde{A}^2}{2m}\tau_3}_{-\hat{X}_2}$$

$$S[\tilde{A}, \tilde{\phi}] = -\mathrm{tr}\left[\ln\left(-\mathcal{G}^{-1}\right)\right] = -\mathrm{tr}\left\{\ln\left[-\mathcal{G}_0^{-1}\left[1 - \mathcal{G}_0(\hat{X}_1 + \hat{X}_2)\right]\right]\right\}$$

$$= \mathrm{const} + \underbrace{\mathrm{tr}[\mathcal{G}_0 \hat{X}_1]}_{\text{1st order } \tilde{A}, \tilde{\phi}} + \mathrm{tr}\underbrace{[\mathcal{G}_0 \hat{X}_2 + \frac{1}{2}\mathcal{G}_0 \hat{X}_1 \mathcal{G}_0 \hat{X}_1]}_{\text{2nd order in } \tilde{A}, \tilde{\phi}}$$

the liner term

$$S^{(1)}[\tilde{A}, \tilde{\phi}] = \frac{1}{V\beta} \sum_{\vec{p}, ip_n} [\mathcal{G}_0(p) \hat{X}_1(p,p)] = \frac{1}{V\beta} \mathrm{tr} \sum_{\vec{p}, ip_0} [\mathcal{G}_0(p)(i\tilde{\phi}_0 \tau_3 + \frac{i}{m}\vec{p}\cdot\tilde{A}_0)]$$

$\langle p | \mathcal{G}_0 | p \rangle \langle p | X_1 | p \rangle$

the zero frequency uniform part

the term propto to $\vec{p}$ should vanish because $\mathcal{G}(p)$ is even.

$$\frac{1}{\beta}\sum_{ip_n} tr[\mathcal{G}(p)\tau_3] = \frac{1}{\beta}\sum_{ip_n}[\mathcal{G}_{11}(p,ip_n) - \mathcal{G}_{22}(p,ip_n)]$$

according to $\mathcal{G}(k,\tau-\tau') = -\begin{pmatrix} \langle|T_\tau C_k(\tau) C_k^\dagger(\tau')|\rangle, & \langle|T_\tau C_k(\tau) C_{-k}(\tau')|\rangle \\ \langle|T_\tau C_{-k}^\dagger(\tau) C_k^\dagger(\tau')|\rangle & \langle|T_\tau C_{-k}^\dagger(\tau) C_{-k}(\tau')|\rangle \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \mathcal{G}(k,ik_n), & \mathcal{F}(k,ik_n) \\ \mathcal{F}^\dagger(k,ik_n), & -\mathcal{G}(-k,-ik_n) \end{pmatrix}$$

$$\Rightarrow \frac{1}{\beta}\sum_{ip_n} tr[\mathcal{G}(p)\tau_3] = \frac{1}{\beta}\sum_{ip_n}\left[\mathcal{G}_{\uparrow\uparrow}(p,ip_n) + \mathcal{G}_{\downarrow\downarrow}(-p,-ip_n)\right] = \langle\psi_\uparrow^\dagger(p)\psi_\uparrow(p)\rangle + \langle\psi_\downarrow^\dagger(p)\psi_\downarrow(p)\rangle$$

$\rightarrow S^{(1)}[\tilde{A}] = iN\phi_0 \leftarrow$ Should be compensated by the background, we can set to zero.

② the first term in the second order

$$tr[\mathcal{G}_0\hat{\chi}_2] = \frac{1}{2m}\frac{1}{V\beta}\sum_{p}\left[\mathcal{G}_{11}(p,ip_n) - \mathcal{G}_{22}(p,ip_n)\right]\tilde{A}^2(q=0)$$

$$= \frac{n}{2m}\int d\tau \int dr\ A^2(r,\tau)$$ — diamagnetic term.

Now let us consider the second term, where $\hat{\chi}_1 = i\tilde{\phi}\tau_3 + \frac{i}{2m}[\nabla, \tilde{A}]_+$

$\frac{1}{2} tr[\mathcal{G}_0 \hat{\chi}_1 \mathcal{G}_0 \hat{\chi}_1]$,

$$= \frac{1}{2V}\frac{1}{\beta}\sum_{p,q} tr[-\mathcal{G}_{0,p-q/2}\tilde{\phi}_q \tau_3 \mathcal{G}_{0,p+q/2}\tilde{\phi}_{-q} + \mathcal{G}_{0,p-q/2}\langle p-q/2|\tfrac{i}{2m}[\nabla,\tilde{A}]_+|p+\tfrac{q}{2}\rangle \mathcal{G}_{0,p+\frac{q}{2}}\langle p+\tfrac{q}{2}|\tfrac{i}{2m}[\nabla,\hat{A}]_+|p-q/2\rangle$$

* we will neglect the dependence of $\mathcal{G}_{0,p+q}$ on "q", because $\tilde{A}$, $\tilde{\phi}$ already contains information of $\nabla\theta$ and $\partial_t\theta$.

* Let us check $\langle p-q/2|\frac{i}{2m}[\nabla,\tilde{A}]_+|p+\frac{q}{2}\rangle = \frac{i}{2m}[i(p+\frac{q}{2})A(q) - (-i(p-q/2))A(q)]$

$$= \frac{-1}{m}\vec{p}\cdot\vec{A}(q)$$

similarly $\langle p+q/2|\frac{i}{2m}[\nabla,\hat{A}]_+|p-q/2\rangle = \frac{-1}{m}\vec{p}\cdot\vec{A}(-q)$

$\Rightarrow$ * Crossing term between $\tilde{\phi}$, $\vec{p}\cdot\tilde{A}$, vanishes as $q\to 0$, because it is odd in momenta.

$$\mathcal{G}_{0,p} = [i\omega_n - (\tau_3\xi_p - \Delta_0\tau_1)]^{-1} = \frac{1}{\omega_n^2+\xi_p^2+\Delta_0^2}[-i\omega_n - \tau_3\xi_p + \tau_1\Delta_0]$$

$$\Rightarrow \frac{1}{2}tr[\mathcal{G}_0\hat{\chi}_1\mathcal{G}_0\hat{\chi}_1] = \frac{1}{2V\beta}tr\left\{\left[\frac{-i\omega_n - \tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}[-\tilde{\phi}_q\tau_3]\frac{-i\omega_n-\tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}[-\tilde{\phi}_{-q}\tau_3]\right\}\right.$$

$$+\frac{1}{2V\beta}\,tr\left\{\frac{-i\omega_n-\tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}\left[\frac{-1}{m}\vec{p}\cdot\vec{A}(q)\right]\frac{-i\omega_n-\tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}\left[\frac{-1}{m}\vec{p}\cdot\vec{A}(-q)\right]\right\}$$

$$=\frac{1}{V\beta}\sum_{p,q}\frac{1}{(\omega_n^2+\lambda_p^2)^2}\left[\tilde{\phi}_q\tilde{\phi}_{-q}\left[-\omega_n^2+\xi_p^2-\Delta_0^2\right]+\frac{(\vec{p}\cdot\vec{A}(q))(\vec{p}\cdot\vec{A}(-q))}{m^2}(-\omega_n^2+\xi_p^2+\Delta_0^2)\right]$$

set $\lambda_p^2=\xi_p^2+\Delta_0^2$

$$=\frac{1}{V\beta}\sum_{p,q}\frac{1}{(\omega_n^2+\lambda_p^2)^2}\left(\tilde{\phi}_q\tilde{\phi}_{-q}(-\omega_n^2+\lambda_p^2-2\Delta_0^2)+\frac{p^2\vec{A}(q)\cdot\vec{A}(-q)}{3m^2}(-\omega_n^2+\lambda_p^2)\right]$$

$\longrightarrow$ translate back to real space of $\tilde{\phi}$, $\tilde{A}$ $\left[\begin{array}{l}\tilde{\phi}=\phi+\partial_\tau\theta\\ \tilde{A}=A-\nabla\theta\end{array}\right]$.

$$S[\hat{A}]=\int d\tau\,dr\left[\underbrace{\frac{1}{V\beta}\sum_p\frac{-\omega_n^2+\lambda_p^2-2\Delta_0^2}{(\omega_n^2+\lambda_p^2)^2}}_{C_1}\tilde{\phi}^2(r,\tau)\right.$$
$$\left.+\underbrace{\left(\frac{n}{2m}-\frac{1}{3m^2}\frac{1}{V\beta}\sum_p\frac{p^2(-\omega_n^2+\lambda_p^2)}{(\omega_n^2+\lambda_p^2)}\right)}_{C_2}\tilde{A}^2(r,\tau)\right]$$

do frequency summation $\leftarrow$ encercise

$$\frac{1}{\beta}\sum_n\frac{-\omega_n^2+\lambda_p^2-2\Delta_0^2}{(\omega_n^2+\lambda_p^2)^2}=-\frac{1}{2\lambda_p}\left[n_F(-\lambda_p)\left(\frac{\Delta_0}{\lambda_p}\right)^2+n_F'(-\lambda_p)\frac{\xi_p^2}{\lambda_p}\right]+(\lambda_p\to-\lambda_p)$$
$$\approx-\Delta_0^2/(2\lambda_p^3)$$

$$\frac{1}{\beta}\sum_n\frac{-\omega_n^2+\lambda_p^2}{(\omega_n^2+\lambda_p^2)^2}=-\beta\left[n_F(\lambda_p)(1-n_F(\lambda_p)\right]$$

$$\Rightarrow \; C_1 = -\frac{1}{V}\sum_P \frac{\Delta^2}{\lambda_P^3} = -\frac{N_0}{2}\int d\xi \, \frac{\Delta_0^2}{(\xi^2+\Delta_0^2)^{3/2}} = -\frac{N_0}{2}\int dx \, \frac{1}{(x^2+1)^{3/2}}$$

$$= -N_0$$

p^2 is peaked at k_F

$$C_2 = \frac{n}{2m} - \frac{N_0}{3m^2} k_F^2 \int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda)) \; .$$

at $T \ll \Delta_0$; $C_2 \simeq \frac{n}{2m}$, i.e. the response is dominated by the diamagnetic term.

at $T > \Delta_0$. $\int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda)) = -\int_{-\infty}^{\infty} d\xi \; \partial_\xi n_F(\xi) = n_F(-\infty) - n_F(\infty)$
$\Rightarrow \lambda = \xi$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1$

$$n_F(\xi) = \frac{1}{e^{\beta\xi}+1} \qquad \partial_\xi n_F(\xi) = \frac{-e^{\beta\xi}\cdot\beta}{(e^{\beta\xi}+1)^2} = \beta \, n_F(1-n_F) \; .$$

$\Rightarrow$ $C_2 = 0$. The diagramtic response should be canceled by the paramagnetic response.

in the middle $\Rightarrow$ $C_2 = \frac{n_s}{2m} = n - \frac{2N_0 k_F^2}{3m}\int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda))$

$$= n - n_s$$

the second term is the normal fluid density.