Lett 13: Electro-magnetic response

1. Linear response

$$H = \int dx \ \psi^{\dagger} \frac{+\hbar}{2m} (-i\hbar\nabla - \frac{e}{c}A)^2 \psi \quad + \quad H_{int}$$

⇒ The term at A's linear order

$$H_1 = \frac{ie\hbar}{2mc} \int \psi^{\dagger} (A\cdot\nabla + \nabla\cdot A) \psi \, dr,$$ perform Fourier transform

$$H_1 = -\frac{e\hbar}{mc} \sum_{kq\sigma} (\vec{k} + \vec{q}/2) \cdot \vec{A}(q) \ C^{\dagger}_{k+q\sigma} C_{k\sigma} .$$

we use Coulumb gauge $\vec{q}\cdot\vec{A}(q) = 0 \Rightarrow H_1 = -\frac{e\hbar}{mc} \sum_{kq} (\vec{k}\cdot\vec{A}(q)) (C^{\dagger}_{k+q\uparrow} C_{k\uparrow} - C^{\dagger}_{-k\downarrow} C_{-k-q\downarrow})$

using $C^{\dagger}_{k\uparrow} = u_k \alpha^{\dagger}_{k\uparrow} - v_k \alpha_{-k\downarrow}$, $C^{\dagger}_{-k\downarrow} = u_k \alpha^{\dagger}_{-k\downarrow} + v_k \alpha_{k\uparrow}$

$$\Rightarrow H_1 = -\frac{e\hbar}{mc} \sum_{kq} [\vec{k}\cdot\vec{A}(q)] \Big\{ [u_{k+q} u_k \alpha^{\dagger}_{k+q\uparrow} \alpha_{k\uparrow} + v_{k+q} v_k \alpha_{-k-q\downarrow} \alpha^{\dagger}_{-k\downarrow}$$
$$- u_{k+q} v_k \alpha^{\dagger}_{k+q\uparrow} \alpha^{\dagger}_{-k\downarrow} - v_{k+q} u_k \alpha_{-k-q\downarrow} \alpha_{k\downarrow}] - [u_{k+q} u_k \alpha^{\dagger}_{-k\downarrow} \alpha_{-k-q\downarrow}$$
$$+ v_k v_{k+q} \alpha_{k\uparrow} \alpha^{\dagger}_{k+q\uparrow} + u_k v_{k+q} \alpha^{\dagger}_{-k\downarrow} \alpha^{\dagger}_{k+q\uparrow} + v_k u_{k+q} \alpha_{k\uparrow} \alpha_{-k-q\downarrow}] \Big\}$$

$$= -\frac{e\hbar}{mc} \sum_{kq} [\vec{k}\cdot\vec{A}(q)] \Big\{ (u_{k+q} u_k + v_{k+q} v_k) (\alpha^{\dagger}_{k+q\uparrow} \alpha_{k\uparrow} - \alpha^{\dagger}_{-k\downarrow} \alpha_{-k-q\downarrow})$$
$$+ (-u_{k+q} v_k + u_k v_{k+q}) [\alpha^{\dagger}_{k+q\uparrow} \alpha^{\dagger}_{-k\downarrow} + \alpha_{k\uparrow} \alpha_{-k-q\downarrow}] \Big\}$$

the second quantization formula of electron current:

$$\vec{j}(r) = \frac{e\hbar}{2mi}\left[\psi^\dagger \nabla\psi - (\nabla\psi^\dagger)\psi\right] - \frac{e^2}{mc}\psi^\dagger A \psi = \frac{\delta H}{\delta A}$$

$$= \vec{j}_1(r) + \vec{j}_2(r)$$

($\vec{j}_1$: paramagnetic current

$\vec{j}_2$: diamagnetic current).

$$j_1(r) = \frac{e\hbar}{2mi}\left(\psi^\dagger(\nabla\psi) - (\nabla\psi^\dagger)\psi\right) = \frac{e\hbar}{m}\sum_{kq}(\vec{k}+\vec{q}/2)\, e^{+iqr}\left[C^\dagger_{k+q\uparrow}C_{k\uparrow} - C^\dagger_{-k\downarrow}C_{-k-q\downarrow}\right]$$

$$j_2(r) = \frac{-e^2}{mc}\psi^\dagger A \psi = \frac{-e^2}{mc}A(r)\,\psi^\dagger(r)\psi(r).$$

Linear response to A's first order, the contribution from $j_2(r)$

is just

$$\langle j_2(r)\rangle = -\frac{e^2}{mc}A(r)\,\langle\Omega|_0\, \psi^\dagger(r)\psi(r)|\Omega\rangle_0 = -\frac{ne^2}{mc}A(r)$$

$$= -\frac{c}{4\pi}\lambda_L^{-2}\vec{A}(r).$$

↘ we only need to keep at 0th order.

diamagnetic current.

Such a term is the same at Normal and SC state.

What's different is the paramagnetic part.

$$\vec{j}_1(r) = \frac{e\hbar}{m}\sum_{kq}(\vec{k}+\vec{q}/2)\,e^{i\vec{q}\vec{r}}\cdot\left\{(u_{k+q}u_k + v_{k+q}v_k)(\alpha^\dagger_{k+q\uparrow}\alpha_{k\uparrow} - \alpha^\dagger_{-k\downarrow}\alpha_{-k-q\downarrow})\right.$$

$$\left. + (-u_{k+q}v_k + u_k v_{k+q})(\alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} + \alpha_{k\uparrow}\alpha_{-k-q\downarrow})\right.$$

Calculation of the paramagnetic current.

$$|\Omega\rangle_1 = |\Omega\rangle_0 + \sum_l{}' \, |l\rangle_0 \frac{{}_0\langle l | H_1 | \Omega\rangle_0}{E_0 - E_l}$$

$$j_1(r) = {}_0\langle \Omega | j_1 | \Omega\rangle_0 + \sum_l{}' \left\{ \frac{{}_0\langle \Omega | j | l\rangle_0 \, {}_0\langle l | H' | \Omega\rangle_0}{E_0 - E_l} \right.$$

$$\left. + \frac{{}_0\langle \Omega | H_1 | l\rangle_0 \, {}_0\langle l | j | \Omega\rangle_0}{E_0 - E_l} \right\},$$ where $|\Omega\rangle_0$ is the BCS ground state is the vaccum of quasi-particles

thus $|l\rangle_0 = \alpha^+_{k+q\uparrow} \alpha^+_{-k\downarrow} |0\rangle$, $E_0 = 0$, $E_l = E_{k+q} + E_k$

$$\langle l | H_1 | \Omega\rangle = -\frac{e\hbar}{mc} (\vec{k}\cdot\vec{A}(q)) [-u_{k+q} v_k + u_k v_{k+q}]$$

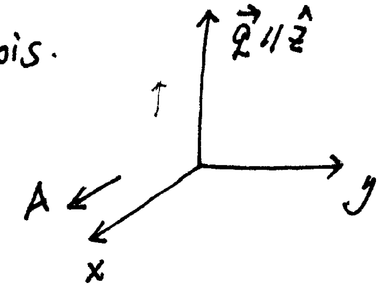
$$\langle l | j_1 | \Omega\rangle = \frac{e\hbar}{m} [(\vec{k}+\vec{q}/2) e^{i\vec{q}\vec{r}}] [-u_{k+q} v_k + u_k v_{k+q}]$$

$$\langle \Omega | j_1 | \Omega\rangle = 0$$

$$\Rightarrow \vec{j}_1(r) = \left(\frac{e\hbar}{m}\right)^2 \frac{1}{c} \sum_{kq} (u_{k+q} v_k - u_k v_{k+q})^2 \frac{(\vec{k}+\vec{q}/2)(\vec{k}\cdot\vec{A}(q))}{\xi_k + \xi_{k+q}} e^{iqr}$$

$$\Rightarrow \boxed{\vec{j}_1(q) = 2 \frac{e^2\hbar^2}{m^2 c} \sum_k \frac{(\vec{k}\cdot\vec{A}(q))\,\vec{k}}{E_{k+q/2} + E_{k-q/2}} (u_{k+q/2} v_{k-q/2} - u_{k-q/2} v_{k+q/2})^2}$$

set $\vec{q}$ along the $\hat{z}$-axis, and $\vec{A}$ along $\hat{x}$-axis.



we first prove $\vec{j}_1(q)$ can be only along x-axis

The system has the symmetry of reflection respect to y-axis, thus $\vec{j}_1(q)$ can only lie in the xz-plane. Let's further prove the $\hat{z}$-ampont is zero.

$$(\vec{k}\cdot\vec{A}(q)) = kA\sin\theta_k\cos\varphi_k \qquad \vec{k} = k[\sin\theta_k\cos\varphi_k, \sin\theta_k\sin\varphi_k, \cos\theta_k]$$

if we take the $\hat{z}$-component, $(\vec{k}\cdot\vec{A}(q))k_z$ contribute the anglular form facter $\sin\theta_k\cos\theta_k\cdot\cos\varphi_k$, however $E_{k+q/2}$, $E_{k-q/2}$, $u_{k\pm q/2}$, $v_{k\pm q/2}$ only depends on $|k\pm q/2| = k^2 + (q/2)^2 \pm kq\cos\theta_k$, has no dependence on φ_k, thus the integral over the azimuthal angle will make the j_z vanish. The only non-vanish component is x-direction, in other words, to be parallel to $\vec{A}$.

$$\Rightarrow \quad \vec{j}_1(q) = \frac{2e^2\hbar^2}{m^2c}\left\{\frac{1}{4\pi}\int_0^{2\pi}\cos^2\phi\, d\phi\int_{-1}^{+1}\sin^2\theta\, d\cos\theta\ \frac{N_0}{2}\int_{-\infty}^{+\infty}d\epsilon\ \frac{(u_+v_- - u_-v_+)^2}{E_+ + E_-}\right\}\vec{A}(q)$$

$$(u_+v_- - u_-u_+)^2 = \frac{1}{2}\left\{\frac{E_+E_- - \epsilon_+\epsilon_- - \Delta^2}{E_+E_-}\right\}$$

$$\Rightarrow \quad \vec{j}_1(q) = \frac{c}{4\pi}\frac{\vec{A}(q)}{\lambda_L^2}\left(\frac{3}{4}\right)\int_{-1}^{1}(1-z^2)\,dz\int_{-\infty}^{+\infty}d\epsilon\left\{\frac{1}{2}\ \frac{E_+E_- - \epsilon_+\epsilon_- - \Delta^2}{E_+E_-(E_+ + E_-)}\right\}$$

$\Rightarrow$ The total response

$$\vec{j}(q) = -\frac{c}{4\pi}\frac{\vec{A}(q)}{\lambda_L^2}\left[1 - \frac{3}{4}\int_{-1}^{1}(1-z^2)\,dz\int_{-\infty}^{+\infty}d\epsilon\ \frac{1}{2}\ \frac{E_+E_- - \epsilon_+\epsilon_- - \Delta^2}{E_+E_-(E_+ + E_-)}\right].$$

where $\epsilon_\pm = \epsilon \pm \frac{1}{2}\hbar q v_F z$.

* at normal state, $\Delta = 0$, $E_\pm = |\epsilon_\pm|$

$$\int_{-\infty}^{+\infty} d\epsilon \; \frac{1}{2} \; \frac{|\epsilon_+||\epsilon_-| - \epsilon_+\epsilon_-}{|\epsilon_+||\epsilon_-|} \; \frac{1}{|\epsilon_+| + |\epsilon_-|} = \int_{-\frac{1}{2} q v_F z}^{\frac{1}{2}\hbar q v_F z} \frac{d\epsilon}{\epsilon_+ - \epsilon_-} \approx 1$$

$$1 - \frac{3}{4}\int_{-1}^{1}(1-z^2)\,dz = 0$$ $\Rightarrow$ paramagnetic and diamagnetic currents cancels.

* for the superconducting states

if $q\xi_0 \ll 1$, where $\xi_0 = \frac{\hbar v_F}{\pi \Delta(0)}$, *

$$E_\pm = \sqrt{(\epsilon_\pm)^2 + \Delta^2} = E \pm \frac{\epsilon}{E}\cdot\left(\frac{1}{2}\hbar q v_F z\right), \quad \epsilon_\pm = \epsilon \pm \frac{1}{2}\hbar q v_F z$$

$$\Rightarrow E_+E_- - \epsilon_+\epsilon_- - \Delta^2 = (E^2 - \epsilon^2 - \Delta^2) - \left[\frac{\epsilon^2}{E^2} - 1\right]\left[\frac{1}{2}\hbar q v_F z\right]^2$$

$$= \frac{\Delta^2}{E^2}\left(\frac{1}{2}\hbar q v_F z\right)^2$$

the integrand goes as $\left(\frac{q v_F \hbar}{\Delta}\right)^2 \ll 1$,

thus j_1 current $\to 0$, j_2 term survives.

$$\vec{j}(q) \approx -\frac{c}{4\pi}\frac{\vec{A}(q)}{\lambda_L^2}$$

if $q\xi_0 \gg 1$, the calculation is much more complicated, and we get pippard form. we will not present it here!

14 : Ginzburg - Landau theory

Definition of the order parameter,

$$\Psi = g \langle \psi_\uparrow(R)\, \psi_\downarrow(R) \rangle = \sum_k \langle a^\dagger_\uparrow(k)\, a^\dagger_\downarrow(-k) \rangle$$

, we expand the G-L free energy as

$$F_{GL}\{\psi(r); T\} = \int dr \left\{ F_0(T) + \alpha(T)\, |\Psi(r)|^2 + \frac{1}{2}\beta(T)\, |\Psi(r)|^4 \right.$$

$$\left. + \gamma(T) \left| \left(\nabla - \frac{ie2}{\hbar c} A(r)\right) \Psi(r) \right|^2 + \frac{1}{8\pi} (\nabla \times A(r))^2, \right.$$

and we need to derive these coefficients from BCS theory. (Ψ is the same as Δ below).

$$H_{MF} = \sum_{k\sigma} E_k \{ \alpha^\dagger_{k\sigma} \alpha_{k\sigma} - 1/2 \} + \sum_k \{ \varepsilon_k - \mu \} + \frac{\Delta^2}{g} \cdot Vol$$

where $E_k = \sqrt{\varepsilon_k^2 + \Delta^2}$.

$$\frac{F}{V} = -\frac{2}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\left[2\cosh \frac{\beta E_k}{2} \right] + \frac{\Delta^2}{g} + const$$

$$\frac{\partial F/V}{\partial \Delta} = -\int \frac{d^3k}{(2\pi)^3} \tanh \frac{\beta E_k}{2} \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}} + \frac{2\Delta}{g}$$

$$= -N(0) \int_0^{\Lambda} d\epsilon \tanh\left[\frac{\beta}{2} \sqrt{\epsilon^2 + \Delta^2} \right] \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}} + \frac{2\Delta}{g}$$

↑ ($N(0)$): density of states of Normal state with two spin-spices.

$$2\alpha = \frac{\partial^2}{\partial\Delta^2}(F/V) = -N(0)\int_0^\Lambda d\epsilon \ \Big[\tanh\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}\ \Big(\frac{1}{\sqrt{\epsilon^2+\Delta^2}} + \frac{(-\frac{1}{2}\Delta)2\Delta}{(\sqrt{\epsilon^2+\Delta^2})^3}\Big)$$

$$+ \frac{\beta}{2}\frac{1}{ch^2\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}\frac{\Delta}{\sqrt{\epsilon^2+\Delta^2}}\frac{\Delta}{\sqrt{\epsilon^2+\Delta^2}} \qquad + \frac{2}{g}$$

near T_c, $\Delta \to 0$,

$$\frac{\partial^2}{\partial\Delta^2}\left(\frac{F}{V}\right) = -N(0)\int_0^\Lambda d\epsilon \ \tanh\frac{\beta\epsilon}{2}/\epsilon \quad + \frac{2}{g}$$

$$= -N(0)\log 1.14\,\beta\Lambda \quad + \frac{2}{g}$$

Forom the self-consistent equation, we have at T_c

$$-N(0)\log 1.14\,\beta_c\Lambda + \frac{2}{g} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial\Delta^2}\frac{F}{V} = -N(0)\log\frac{\beta_c\Lambda}{\beta\Lambda} = +N(0)\log\frac{T}{T_c}$$

$$= -N(0)\left[1-\frac{T}{T_c}\right] \quad \text{at } T\to T_c.$$

$$\Rightarrow \boxed{\alpha = \frac{1}{2}N(0)\left[\frac{T}{T_c}-1\right]}$$

now let us determine β, we calculate

$$\frac{\partial}{\partial(\Delta^2)}\frac{\partial^2}{\partial\Delta^2}\left(\frac{F}{V}\right) = 3\beta$$

$$\frac{\partial}{\partial\Delta^2}\left[\frac{\partial^2 F/V}{\partial\Delta^2}\right] = \frac{\partial}{\partial\Delta^2}\left[-N(0)\int_0^\Lambda d\epsilon \tanh\frac{\beta\sqrt{\epsilon^2+\Delta^2}}{2}\left(\frac{1}{\sqrt{\epsilon^2+\Delta^2}}-\frac{\Delta^2}{(\sqrt{\epsilon^2+\Delta^2})^3}\right)\right.$$

$$\left. + \frac{\beta}{2}\frac{1}{ch^2\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}\frac{\Delta^2}{\epsilon^2+\Delta^2} + \frac{2}{g}\right]\Bigg|_{\Delta^2=0}$$

$$= -N(0)\int_0^\Lambda d\epsilon \left\{\frac{\frac{\beta}{2}}{ch^2\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}\frac{1}{2\sqrt{\epsilon^2+\Delta^2}}\left(\frac{1}{\sqrt{\epsilon^2+\Delta^2}}-\frac{\Delta^2}{(\epsilon^2+\Delta^2)^{3/2}}\right)\right|_{\Delta^2=0}$$

$$+\tanh\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}\left(\frac{-1/2}{(\epsilon^2+\Delta^2)^{3/2}}-\frac{1}{(\epsilon^2+\Delta^2)^{3/2}}-\frac{\Delta^2(-3/2)}{(\epsilon^2+\Delta^2)^{5/2}}\right)\Bigg|_{\Delta^2=0}$$

$$+\frac{\beta}{2}\frac{(-2)\,sh\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}{ch^3\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}\frac{\beta}{2}\frac{1}{2\sqrt{\epsilon^2+\Delta^2}}\frac{\Delta^2}{\epsilon^2+\Delta^2}\Bigg|_{\Delta^2=0}$$

$$+\frac{\beta}{2}\frac{1}{ch^2\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}}\left(\frac{1}{\epsilon^2+\Delta^2}-\frac{\Delta^2}{(\epsilon^2+\Delta^2)^2}\right)\Bigg|_{\Delta^2=0}$$

$$= -N(0)\int_0^\Lambda d\epsilon\left[\frac{\frac{\beta}{2}}{ch^2\frac{\beta}{2}\epsilon}\frac{1}{2\epsilon^2}-\frac{3}{2}\tanh\frac{\beta\epsilon}{2}\frac{1}{\epsilon^3}+\frac{\beta}{2}\frac{1}{ch^2\frac{\beta}{2}\epsilon}\frac{1}{\epsilon^2}\right]$$

$$= -N(0)\frac{3}{2}\int_0^\Lambda d\epsilon\left\{\frac{\beta}{2\epsilon^2}\frac{1}{ch^2\frac{\beta}{2}\epsilon}-\frac{1}{\epsilon^3}\tanh\frac{\beta}{2}\epsilon\right\}$$

$$= -N(0)\frac{3}{2}\frac{\beta^2}{8}\int_0^{\beta\Lambda/2}dx\left[\frac{1}{x^2}\frac{1}{ch^2 x}-\frac{1}{x^3}\tanh x\right]$$

$$= -\frac{3}{2}N(0)\frac{\beta^2}{8}\int_0^{+\infty}dx\; x^{-1}\frac{d}{dx}\left[(\tanh x)/x\right] \quad \leftarrow \frac{-7}{\pi^2}\zeta(3)$$

$$= \frac{3}{2}N(0)\frac{7}{8\pi^2}\zeta(3)\frac{1}{(kT_c)^2} \Rightarrow \boxed{\beta = \frac{1}{2}N(0)\frac{7}{8\pi^2}\zeta(3)\frac{1}{(kT_c)^2}}$$

how to calculate $\gamma (\nabla \Psi(r))^2$? suppose we creat a

$$\Psi(r) = \Delta e^{i\phi(r)} \Rightarrow \Delta F = \frac{1}{2}\rho_s(T) v_s^2 = \frac{1}{2}\rho_s(T)\left(\frac{\hbar}{2m}\nabla\phi\right)^2$$

$$= \gamma \Delta^2(T) (\nabla\phi)^2$$

$$\Rightarrow \gamma(T) = \frac{\hbar^2 \rho_s(T)}{8m^2 \Delta^2(T)}$$

we have evaluated the superfluid density $\rho_s(T) = \rho(1 - Y(\frac{T}{T_c}))$, where $\rho = nm$ is the total mass density. Near T_c, Yosida function can be approximated $1 - Y(T/T_c) \approx \frac{7}{4}\zeta(3)[\Delta^2(T)/\pi^2 k_B^2 T_c^2)$

$$\Rightarrow \boxed{\gamma = \frac{n\hbar^2}{4m} \frac{7\zeta(3)}{8\pi^2 k_B^2 T_c^2}}$$

Summary: in the clean superconductor, $\Psi(r)$ is normalized as the gap function, then:

$$\gamma = \frac{n\hbar^2}{2m}\frac{7\zeta(3)}{8\pi^2 k_B^2 T_c^2}, \quad \alpha = \frac{N(0)}{2}\left(\frac{T}{T_c} - 1\right), \quad \beta = \frac{N(0)}{2}\frac{7}{8\pi^2}\zeta(3)\frac{1}{(k_B T_c)^2}.$$

Correlation length / healing length

$$\xi(T) = \left(\frac{\gamma}{\alpha(T)}\right)^{1/2} = \left[\frac{n\hbar^2}{4m N(0)} \frac{7\zeta(3)}{4\pi^2 (k_B T_c)^2}\right]^{1/2} \left(1 - \frac{T}{T_c}\right)^{-1/2}$$

$$\approx \left(\frac{7\zeta(3)}{48}\right)^{1/2} \frac{\hbar v_F}{\pi k_B T_c}\left(1 - \frac{T}{T_c}\right)^{-1/2} = 0.74\,\xi_0 \left(1 - \frac{T}{T_c}\right)^{-1/2}$$

$\xi_0 = 0.18\hbar v_F / kT_c \pi$ is the Pippard coherence length, which is roughly the "radius of Cooper pairs".

Physical meaning of $\xi(T)$: The equilibrium value of $\Psi(r)$

$$-\alpha \Psi(r) + \beta |\Psi|^2 \Psi(r) = 0 \Rightarrow \Psi(r) = \psi_0 = \left(\frac{-\alpha}{\beta}\right)^{1/2}.$$

a small deviation $\psi(r) = \psi_0 + \delta\psi(r) \Rightarrow -2\alpha\,\delta\psi(r) - \gamma \nabla^2 \delta\psi(r) = 0$

$$\Rightarrow \delta\psi \sim \text{const}\; e^{-\sqrt{2}(x/\xi(T))},$$

$\xi(T)$ is the healing length.

at $T \to T_c$, $\xi(T)$ divergences, $\Psi(r)$ varies at the scale of $\xi(T)$, which is much longer than the Cooper pair size of ξ_0, which is precisely the condition of the validity of G-L free energy.

§ Dirty Superconductors

If the mean free path $l \lesssim \xi_0$, we say the superconductor is in the dirty region. Amazingly, superconductivity does not change much due to the Anderson theorem.

$$H = H_0 + H_{int}, \quad H_0 = \sum_i \left(\frac{p_i^2}{2m} + U(r_i) - \mu \right)$$

Suppose we have found the single-particle eigenstates

$$H_0 \varphi_n(r) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(r)\right] \varphi_n(r) = \mathcal{E}_n \varphi_n(r),$$ due to the time-reversal symmetry, the eigenstates can always be paired

$\varphi_{n\uparrow}$, $\varphi^*_{n\downarrow} \to \varphi_{-n,\downarrow}$, and we can try the BCS function of $\Psi = \prod_n (u_n + v_n a^+_{n\uparrow} a^+_{-n\downarrow}) |vac\rangle$, the only difference is ~~the~~ to use $\varphi_n(r)$ to replace plane wave state. Similarly the gap equation

$$\boxed{\Delta_m = -\sum_n V_{mn} \frac{\Delta_n}{2E_n} \tanh\left(\frac{\beta}{2}E_n\right)},$$

$$V_{mn} = \int dr dr' \, \varphi^*_m(r) \varphi^*_{-m}(r') \, \varphi_{-n}(r') \varphi_n(r) \, V(r-r').$$

$$\to \quad \Delta = (V_0)_a (N(0))_a \int_{-\epsilon_c}^{\epsilon_c} \frac{\Delta}{2E} \tanh \frac{\beta}{2} E \, d\epsilon,$$

From thermodynamics point of view, a dirty superconductor is not very much different from a pure one. But there are other differences.

<u>Cooper pair size:</u>

define $F(r_1, r_2) = \langle \psi_\downarrow(r_1) \psi_\uparrow(r_2) \rangle = F(R, \rho)$

where $R = \frac{r_1 + r_2}{2}$ $\quad \rho = r_1 - r_2$,

$$\boxed{F(r_1, r_2) = \sum_n \frac{\Delta_n}{2E_n} \tanh \frac{\beta}{2} E_n \, \varphi^*_n(r_1) \varphi_n(r_2)}$$

if we set $r_1 \to r_2$, $F(R, \rho=0) = \sum_n \frac{\Delta_n}{2E_n} \tanh \frac{\beta}{2} E_n |\varphi_n(R)|^2$ is just the gap equation.

But as we increase $\rho = r_1 - r_2$, $F(R, \rho)$ describes the internal structure of the Cooper pair, it's a sum over wave pocket close to the Fermi surface. at $\rho = 0$, each term of $\varphi_n^*(r_1)\varphi_n(r)$ has the same phase over all "n"; as ρ increases, $\varphi_n^*(r_1)\varphi_n(r_2)$ is complex number. and has phase dispersion for different n. If the variation range $\ll 2\pi$, we get constructive interference; otherwise we get destructive interference. Basically, the Cooper pair size can be determined roughly by the extention $\rho = |r_1 - r_2|$, at which the phases $\varphi_n^*(r_1)\varphi_n(r_2)$ varying $\sim 2\pi$ for different n.

Semiclassicaly, a wave pocket ~~starting~~ at r_1 with an energy distribut ΔE, ~~we wi~~ will have a dispersion of 2π (dephase), at the time $\sim \frac{h}{\Delta E}$, During this amont of time, particle move $V_F \cdot \frac{h}{\Delta E}$, which is the Cooper pair size. ΔE is the gap energy at zero ~~temperature~~, while ΔE is around kT_c around T_c. Thus Roughly $\xi \sim \frac{V_F h}{\Delta}$ or $\frac{V_F \hbar}{k_B T_c}$. So far, this picture is for clean system. In dirty superconductor, electrons move diffusively, thus $\xi \sim (D\Delta t)^{1/2}$ and $D = v_F l$.

thus $\xi \sim (v_F l \Delta t)^{1/2} = (v_F l \frac{\hbar}{\Delta E})^{1/2} = (\xi_0 l)^{1/2}$, thus Cooper pair size shrinks by a factor $(\frac{\xi_0}{l_0})^{-1/2}$.

How the coefficients of G-L equation change? α and β are related to the bulk thermodynamic quantities, which should not change much. But the coefficient of γ describe the bending energy. If the order parameter strongly varies in the size of Cooperpair, we will lose the condensation energy. In the dirty limit, the size of Cooper pair shinks, thus γ is also suppressed.

$$\gamma_{dirty} \cdot \xi_{dirty}^{-2} = \gamma_0 \xi_0^{-2} \Rightarrow \gamma_{dirty} = \gamma_0 \frac{\xi_{dirty}^2}{\xi_0^2} = \gamma_0 \frac{l}{\xi_0}.$$

thus

$$\boxed{\xi_{dirty}(T) = (\xi_0 l)^{1/2} (1 - T/T_c)^{-1/2}.}$$