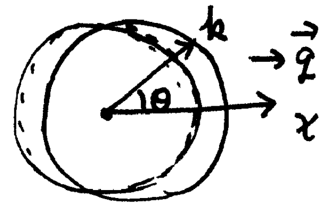
Sloutin to Mid term

1 a) $$\chi_0(q,\omega) = 2\int \frac{d^2k}{(2\pi)^2} \frac{n_{k+q/2} - n_{k-q/2}}{\omega - (\varepsilon_{k+q/2} - \varepsilon_{k-q/2}) + i\eta}$$



set the direction of q along the x-axis

$$n_{k+q/2} = n_k + \frac{\partial n}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial \vec{k}} \cdot \frac{\vec{q}}{2} = n_k + \frac{\partial n}{\partial \varepsilon} \vec{v}_k \cdot \frac{\vec{q}}{2}$$

$$n_{k-q/2} = n_k - \frac{\partial n}{\partial \varepsilon} \vec{v}_k \cdot \frac{\vec{q}}{2}$$

$$\varepsilon_{k+q/2} - \varepsilon_{k-q/2} = \vec{v}_k \cdot \vec{q}$$

$$\Rightarrow \chi_0(q,\omega) = 2\int \frac{d^2k}{(2\pi)^2} \frac{\left(\frac{\partial n}{\partial \varepsilon}\right) \vec{v}_k \cdot \vec{q}}{\omega - \vec{v}_k \cdot \vec{q} + i\eta} = 2\int \frac{d^2k}{(2\pi)^2} \frac{-\delta(\cancel{\varepsilon} - \varepsilon_F)^{(b)}\, v_F q \cos\theta}{\omega - v_F q\cos\theta + i\eta}$$

$$= N(0) \int \frac{d\theta}{2\pi} \frac{-v_F q \cos\theta}{\omega - v_F q \cos\theta + i\eta} = N(0) \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{-\cos\theta}{s - \cos\theta + i\eta}$$

b) $$\frac{1}{x \pm i\eta} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

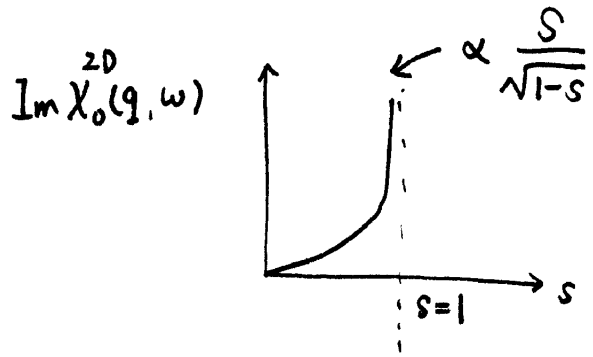
$$\mathrm{Im}\,\chi_0(q,\omega) = N(0) \int_0^{2\pi} \frac{d\theta}{2\pi} (-\cos\theta)(-)\pi\,\delta(s - \cos\theta)$$

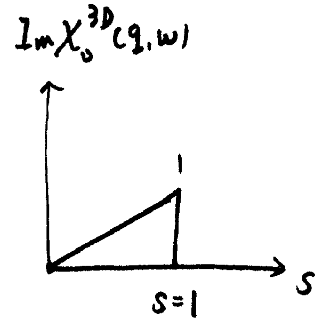
$$= N(0) \cdot \left[\frac{\pi}{2\pi} \cdot s \Big/ \left|\frac{d\cos\theta}{d\theta}\right|_{\theta = \cos^{-1} s}\right] \quad (\text{if } 0 < s < 1)$$

$$= N(0) \frac{s}{2} \Big/ (\sin\theta)\Big|_{\theta = \cos^{-1} s} = \frac{N(0)}{\boxed{}} \frac{s}{\sqrt{1 - s^2}} \quad (\text{for } 0 < s < 1)$$

there are two solutions at theta=\pm arccos s

$$\mathrm{Im}\,\chi_0(q,\omega) = 0 \quad (\text{for } s > 1)$$





$\mathrm{Im}\,\chi_0^{2D}(q,\omega)$ has a divergence at $s=1$, where at 3D, there's no divergence.

This divergence occurs at $s\sim 1$, i.e arond $\theta=0$ in the integrand.

This is because the different measures in the intergrand, between 2D $\int d\theta$ and 3D $\int \sin\theta d\theta$. The phase space weight around $\theta=0$ at 2D is enhanced.

c) in the limit $s\gg 1$, we calculate the real part

$$\mathrm{Re}\,\chi^{2D}(q,\omega) = N_0\int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{-\cos\theta/s}{1-\cos\theta/s} = N_0\int_0\frac{d\theta}{2\pi}\left(-\frac{\cos\theta}{s}\right)\left(1+\frac{\cos\theta}{s}+\cdots\right)$$

$$= -\frac{N_0}{2s^2}$$

$$\varepsilon(q,\omega) = 1-\frac{2\pi e^2}{q}\,\frac{N_0}{2s^2} = 0 \Rightarrow \text{plasmon frequency at 2D}$$

$$1 = \frac{\pi e^2 N_0}{q}\cdot\frac{v_F^2 q^2}{\omega^2} \Rightarrow \omega = \sqrt{\pi e^2 N_0}\; v_F\sqrt{q}$$

$$2D\quad N_0 = \frac{m}{\hbar^2\pi}\,,\quad k_F = (2\pi n)^{1/2}$$

$$\Rightarrow \omega = \left(\frac{2\pi n e^2}{m}\right)^{1/2}\sqrt{q} = \left(\frac{4\pi\hbar^2}{2m}\,n\,e^2 k_F\right)^{1/2}\sqrt{q/k_F}$$

define $\pi r_s^2 = 1/n$, $\quad k_F = \left(2\pi \frac{1}{\pi r_s^2}\right)^{1/2} = \frac{\sqrt{2}}{r_s}$

$$\omega = 2\left(\frac{a_0}{r_s}\right)^{3/2}\left(\frac{\hbar^2}{2m}\frac{1}{a_0^2}\cdot\frac{e^2}{a_0}\right)^{1/2}\sqrt{q/k_F}$$

where a_0 is Bohr-radius.

$$\frac{e^2}{a_0} = \frac{\hbar^2}{m}\frac{1}{a_0^2} = E_{Rg}$$

d): since $\omega \propto \sqrt{q}$

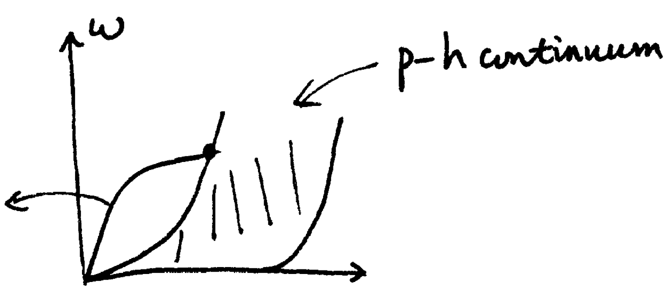
$$E(T) = \int \frac{d^2q}{(2\pi)^2}\frac{\hbar\omega}{e^{\hbar\omega/k T}-1} = \int_0^{\text{cut off}} d\omega\, g(\omega)\frac{\hbar\omega}{e^{\hbar\omega/k T}-1}$$

$$\frac{2\pi q\, dq}{(2\pi)^2} = g(\omega)d\omega \Rightarrow g(\omega) \propto q\frac{dq}{d\omega} \propto \omega^3$$

$$E(T) \propto \int_0^{\text{cut off}} d\omega\, \omega^3 \frac{\hbar\omega}{e^{\hbar\omega/k T}-1} \propto T^5 \int_0^{\text{cut off}/k T} dx \frac{x^4}{e^x-1}$$

$$\Rightarrow C_V \propto T^4$$

The cut-off depends on the location of the plasmon going into the continuum.



2D plasmon freq.

e): Now we have two planes, we consider the in-phase/out of phase plasmon modes. Let's introduce the Fourier components for bilayer.

$k_z = 0$, or π/d

$$v(q,\ k_z=0) = \int d^2r \ \frac{e^2}{r} e^{iqr} + \frac{e^2}{|\vec{r}+\hat{z}d|} e^{i\vec{q}\cdot\vec{r}} = \frac{2\pi e^2}{q}(1+e^{-qd})$$

$$v(q,\ k_z=\frac{\pi}{d}) = \int d^2r \ \frac{e^2}{r} e^{i\vec{q}\cdot\vec{r}} + \frac{e^2}{|\vec{r}+\hat{z}d|} e^{i\vec{q}\cdot\vec{r}+i\pi} = \frac{2\pi e^2}{q^2}(1-e^{-qd})$$

we have two disconnected Fermi surfaces, thus

$$\chi(q0;\omega) = \chi(q\tfrac{\pi}{d};\omega) = \frac{-1}{2} N(0) \frac{1}{S^2}$$ where $N(0)$ is the density of states of a single sheet.

=

$$\varepsilon(q0;\omega) = 1 - \frac{2\pi e^2}{q}(1+e^{-qd}) \ \frac{1}{2}N(0)\frac{1}{S^2} \approx 1 - \frac{2\pi e^2}{q} N(0)\frac{1}{S^2}$$

$\Rightarrow$ $\omega_{\text{in-phase}} \approx \sqrt{2}\ \omega_{\text{2D-plasmon; one plane}}$.

$$\varepsilon(q,\pi/d;\omega) = 1 - \frac{2\pi e^2}{q}(1-e^{-qd}) \ \frac{1}{2}N(0)\frac{1}{S^2} \approx 1 - 2\pi e^2 d\, N(0)\frac{1}{S^2}$$

$\Rightarrow$ $\omega_{\text{out-plase}} = \sqrt{\pi e^2 d N(0)}\ v_F q$,

(this is kind of exciton mode - phonon).

2 a) $\delta E^{(1)} = \sum \varepsilon_p \delta n_{p_\sigma} = V \int d\Omega \int \frac{p^2 dp}{(2\pi)^3} \, \varepsilon_p \, \delta n_\sigma(p, \hat{\Omega}_p)$

$$\int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \int_0^{\delta p_F} \frac{p_F^2 \, dp}{(2\pi)^3} v_F \, \delta p = \frac{p_F^2}{(2\pi)^3} \frac{v_F}{2} (\delta p_F)^2$$

$$\int \frac{p^2 dp}{(2\pi)^3} \, \delta n(p, \hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} \delta p_F = \delta n(\Omega_p)$$

$$\Rightarrow \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \frac{v_F}{2} \left[\delta n(\Omega_p)\right]^2 \Big/ \frac{p_F^2}{(2\pi)^3} = \frac{(\delta n(\Omega_p))^2}{\frac{p_F^2}{(2\pi)^3} \frac{2}{v_F}} = 4\pi N^{-1}(0) (\delta n(\Omega_p))^2$$

$$\frac{\delta E^{(1)}}{V} = N^{-1}(0) \int d\Omega \left[\delta n_\uparrow(\Omega_p)\right]^2 + \left[\delta n_\downarrow(\Omega_p)\right]^2 = 2\pi N^{-1}(0) \sum_{\ell m} \left\{ |\delta n^s_{\ell m}|^2 + |\delta n^a_{\ell m}|^2 \right\}$$

b) $\delta E^{(2)} = \frac{1}{2V} \sum_p \sum_{p'} f_{\sigma\sigma'}(\hat{p}\hat{p}') \, \delta n_{p_\sigma} \delta n_{p'\sigma'} = \frac{V}{2} \int d\Omega_p \int d\Omega_{p'} \, f_{\sigma\sigma'}(\hat{p}\hat{p}') \, \overbrace{\delta n_\sigma(\Omega_p) \delta n_{\sigma'}(\Omega_{p'})}$

$$= \frac{V}{2} N^{-1}(0) \int d\Omega_p \int d\Omega_{p'} \left\{ \left[\sum_{\ell m} F_\ell^s \cdot \frac{4\pi}{2\ell+1} Y^*_{\ell m}(\Omega_p) Y_{\ell m}(\Omega_{p'}) \right] \right.$$

$$\left. \left[\sum_{\ell_1 m_1} Y_{\ell_1 m_1}(\Omega_p) \, \delta n^s_{\ell_1 m_1} \right] \left[\sum_{\ell_2 m_2} Y_{\ell_2 m_2}(\Omega_{p'}) \, \delta n^s_{\ell_2 m_2} \right] + (s \to a) \right\}$$

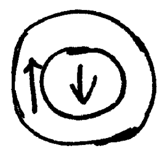
where $F_{\sigma\sigma'} = F^s + F^a \sigma\sigma'$, $\delta n_{s,a} = \delta n_\uparrow \pm \delta n_\downarrow$

$$= \frac{1}{2} V N^{-1}(0) \left[\sum_{\ell m} F_\ell^s \cdot \frac{4\pi}{2\ell+1} \, \delta n^{*(s)}_{\ell m} \delta n^{(s)}_{\ell m} + (s \to a) \right]$$

c) add ~~together~~ $\Rightarrow$

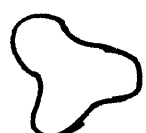
$$\delta E/V = 2\pi N^{-1}(0) \sum_{lm} \left\{ \left(1 + \frac{F_l^s}{2l+1}\right) |\delta n^s_{lm}|^2 + \left(1 + \frac{F_l^a}{2l+1}\right) |\delta n^a_{lm}|^2 \right\}$$

Thus for each channel $F_l^{s,a}$, instability occurs, which can be stabilized by higher order terms

In the channel of F_0^a. $\rightarrow$ Ferromagnetism  .

In the channels of F_l^s $(l>1)$. $\Rightarrow$ anisotropic distortion

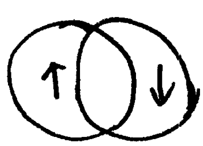
F_2^s



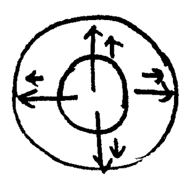
F_3^s

. . .

For the channels of F_l^a $(l \geq 1)$ $\Rightarrow$ more subtle.

F_1^a 

anisotropic phase



isotropic phase

$\vec{S}(k) \parallel \pm \vec{k}$

3: a) The classic Neel states

$$\begin{matrix} A & B & A & B & & \\ \uparrow & \downarrow & \uparrow & \downarrow & \uparrow\downarrow & \cdots \end{matrix}$$

define H-P bosons for A sites $S_A^+ = \sqrt{2s - a^+a}\ a$, $S_A^- = \sqrt{2s-a^+a}\ a^+$, $S_A^z = S_A - a^+a$

B sites $S_B^+ = b^+\sqrt{2s-b^+b}$, $S_B^- = \sqrt{2s-b^+b}\ b$, $S_B^z = b^+b - S_B$

plug in the Hamiltonian and kepp to quadrtic term

$$H = -2NzJ S_a S_b + 2zJ \sum_k \{(S_b a_k^+ a_k + S_a b_k^+ b_k) + \sqrt{S_a S_b}\,\gamma_k (a_k^+ b_k^+ + a_k b_k)\}$$

$\gamma_k = \frac{1}{z} \sum_{\vec{\delta}} e^{i\vec{k}\cdot\vec{\delta}}$, $\vec{\delta}$ is the displacement vector to the nearest neighbours.

$$\Rightarrow \quad H_k = [a_k^+ \ \ b_{-k}^+] \begin{bmatrix} S_b & \sqrt{S_a S_b}\,\gamma_k \\ \sqrt{S_a S_b}\,\gamma_k & S_a \end{bmatrix} \begin{pmatrix} a_k \\ b_{-k}^+ \end{pmatrix}$$

define transformation $\begin{bmatrix} a_k \\ b_{-k} \end{bmatrix} = \begin{bmatrix} ch\theta_k & -sh\theta_k \\ -sh\theta_k & ch\theta_k \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_{-k}^+ \end{bmatrix}$

$$\rightarrow \quad H_k = [\alpha_k^+, \beta_{-k}] \begin{bmatrix} ch\theta_k & -sh\theta_k \\ -sh\theta_k & ch\theta_k \end{bmatrix} \begin{bmatrix} S_b & \sqrt{S_a S_b}\,\gamma_k \\ \sqrt{S_a S_b}\,\gamma_k & S_a \end{bmatrix} \begin{bmatrix} ch\theta_k & -sh\theta_k \\ -sh\theta_k & ch\theta_k \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_{-k}^+ \end{bmatrix}$$

$$= [\alpha_k^+ \ \beta_{-k}] \begin{bmatrix} S_b ch^2\theta + S_a sh^2\theta - \sqrt{S_a S_b}\,\gamma_k sh2\theta, & -\frac{S_a+S_b}{2} sh2\theta + \sqrt{S_a S_b}\,\gamma_k ch2\theta \\ \cdot\cdot & S_a ch^2\theta + S_b sh^2\theta - \sqrt{S_a S_b}\,\gamma_k sh2\theta \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_{-k}^+ \end{bmatrix}$$

we choose $-\frac{S_a+S_b}{2} sh2\theta_k + \sqrt{S_a S_b}\,\gamma_k ch2\theta_k = 0$

$$\tanh 2\theta_k = \frac{2\sqrt{S_a S_b}\,\gamma_k}{(S_a+S_b)} \quad \Rightarrow \quad sh2\theta_k = \frac{2\sqrt{S_a S_b}\,\gamma_k}{\sqrt{(S_a+S_b)^2 - 4 S_a S_b \gamma_k^2}}$$

$$= \frac{2\sqrt{S_a S_b}\,\gamma_k}{\sqrt{(S_a-S_b)^2 + 4 S_a S_b (1-\gamma_k^2)}}$$

$$ch2\theta_k = \frac{S_a + S_b}{\sqrt{(S_a - S_b)^2 + 4S_aS_b(1-\gamma_k^2)}} .$$

plug in the values of $sh2\theta_k$ and $ch2\theta_k \Rightarrow$

the diagnal terms $\dfrac{\omega_\pm(k)}{2zJ} = \dfrac{S_a + S_b}{2} ch2\theta - \sqrt{S_aS_b}\,\gamma_k\, sh2\theta \pm \dfrac{S_a - S_b}{2}$

$$\Rightarrow \frac{\omega_\pm(k)}{zJ} = \frac{(S_a + S_b)^2 - 2\times 2S_aS_b\gamma_k^2 \cdots}{\sqrt{(S_a - S_b)^2 + 4S_aS_b(1-\gamma_k^2)}} \pm (S_a - S_b)$$

$$= \left[(S_a - S_b)^2 + 4S_aS_b(1-\gamma_k^2)\right]^{1/2} \pm (S_a - S_b)$$

expand it aroud $k \to 0 \Rightarrow \gamma_k^2 = \frac{1}{z}\Sigma(k\cdot\delta)^2 = a^2k^2$

$$\Rightarrow \hbar\omega_k^\pm = \begin{cases} 2zJ(S_a - S_b) + 4Ja^2\dfrac{S_aS_b}{S_a - S_b}k^2 \\ 4Ja^2\dfrac{S_aS_b}{S_a - S_b}k^2 \end{cases}$$

b) The quantum fluctuations are the number of H-P bosons

$$\Delta S_A^z = \frac{2}{N}{\sum_k}'\langle a_k^\dagger a_k\rangle = \frac{2}{N}{\sum_k}'\langle(ch\theta\,\alpha_k^\dagger - sh\theta\,\beta_{-k})(ch\theta\,\alpha_k - sh\theta\,\beta_{-k}^\dagger)\rangle$$ (only sum half of BZ)

$$= \frac{2}{N}{\sum_k}' sh^2\theta_k\langle\beta_{-k}\beta_{-k}^\dagger\rangle \to \frac{2}{N}{\sum_k}'\frac{ch2\theta_k - 1}{2}$$

$$= \frac{1}{N}{\sum_k}'\left[\frac{S_a + S_b}{\sqrt{(S_a+S_b)^2 - 4S_aS_b\gamma_k^2}}\right] - 1$$

$$= \frac{1}{2} \int_{FBZ} \frac{d^2 k}{(2\pi)^2} \left[\frac{1}{\sqrt{1 - c^2 \gamma_k^2}} - 1 \right] \qquad c^2 = \frac{4 S_a S_b}{(S_a + S_b)^2}$$

Similarly, we have the same result for

ΔS_B^z.