Itinearant Ferromagnetism (spin-wave

① Define ferromagnetic order parameter $S_\mu(\vec{r}) = \psi^\dagger_\alpha(x)\,\sigma^\mu_{\alpha\beta}\,\psi_\alpha(x)$

and consider the Hamiltonian

$$\boxed{H = \int d^3\vec{r}\;\psi^\dagger_\alpha(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2-\mu\right)\psi_\alpha(\vec{r}) + \frac{1}{2}\int dr\,dr'\,f(r-r')\,\vec{S}(r)\cdot\vec{S}(r')}$$

Define the H-S field

$$n_\mu(\vec{r}) = -\int dr\; f(r-r')\,S_\mu(r')$$

$$\Rightarrow Z = \int D\psi^\dagger D\psi\;\exp\Big[-\int_0^\beta d\tau\Big[\int dr\;\psi^\dagger_\alpha(\vec{r})\Big(\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu\Big)\psi_\alpha(\vec{r})$$

$$+\frac{1}{2}\int dr\,dr'\,f(r-r')\;\vec{S}(r)\cdot\vec{S}(r')\Big]$$

$$=\int D\psi^\dagger D\psi\;D\vec{n}\;\exp\Big[\frac{1}{2}\int_0^\beta d\tau\int dr\,dr'\;f^{-1}(r-r')\,n_i(r)\,n_i(r')$$

$$\cdot\exp\Big[-\int_0^\beta d\tau\int dr\Big\{\psi^\dagger_\sigma(r)\Big[\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu\Big]\psi_\sigma(r) - \psi^\dagger_\alpha(r)\,\vec{n}\cdot\vec{\sigma}_{\alpha\beta}\,\psi_\beta(r)\Big\}$$

f^{-1} is defined as

$$\int dr\; f(r_1-r)\,f^{-1}(r-r_2) = \delta(r_1-r_2)$$

Now integrate over ψ, we have $Z = \int D\vec{n}\; e^{-S_{eff}(\vec{n})}$

with $$\boxed{S_{eff}(\vec{n}) = -\frac{1}{2}\int_0^\beta d\tau\int dr\,dr'\;f^{-1}(r-r')\,\vec{n}(r)\cdot\vec{n}(r') \pm \mathrm{Tr}\Big[\ln\Big[\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu-\vec{n}\cdot\vec{\sigma}\Big]\Big]}$$

expand $\vec{n} = \bar{n}_\mu + \delta n_\mu(\vec{r},\tau)$

$$\ln\left[-\left(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu - \vec{\bar{n}}\cdot\vec{\sigma}\right) + \delta n_\mu\cdot\vec{\sigma}\right] = \ln\left[G_0^{-1}(\bar{n})\left(1 + G_0(\bar{n})\,\delta n\cdot\vec{\sigma}\right)\right]$$

add a minus sign — a const in the Seff because $\ln(-a\ldots] = \ln(-) + \ln a$

$$\Rightarrow \quad S_{eff}(\vec{n}) = \frac{-1}{2}\int_0^\beta d\tau \int dr\,dr'\, f^{-1}(r-r')\,(\bar{n}_i + \delta n_i(r))\,(\bar{n}_i + \delta n_i(r'))$$

$$- \mathrm{Tr}\ln[G_0^{-1}(\bar{n})] - \mathrm{Tr}\ln\left(1 + G_0(\bar{n})\,\delta\vec{n}\cdot\vec{\sigma}\right)$$

Keep to 2nd order

$$- \mathrm{Tr}\left[G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}\right] + \frac{1}{2}\mathrm{Tr}\left[G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}\; G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}\right]$$

$$\delta S_{eff}(\vec{n}) = -\int_0^\beta d\tau \int dr\,dr'\, f^{-1}(r-r')\,\bar{n}_i\,\delta n_i(r';\tau)$$

$$-\int_0^\beta d\tau \int dr\, \langle \vec{r},\alpha\tau|\, G_0(\bar{n})\,\sigma^i\,|\vec{r}\beta\tau\rangle\,\delta n_i(\vec{r};\tau)$$

$$-\frac{1}{2}\int_0^\beta d\tau \int dr\,dr'\, f^{-1}(r-r')\,\delta n_i(r)\,\delta n_i(r')$$

$$+\int d\tau\,d\tau' \int dr\,dr'\, \langle r\alpha\tau|\, G_0(\bar{n})\,\delta n_i(r)\,\sigma^i\,|r'\beta\tau'\rangle$$

$$\langle r'\beta\tau'|\, G_0(\bar{n})\,\delta n_j(r)\,\sigma^j\,|r\alpha\tau\rangle$$

Saddle point $\Rightarrow \dfrac{\delta S_{eff}}{\delta n(\vec{r}',\tau')} = 0$ plug in $\dfrac{\delta n(\vec{r},\tau)}{\delta n(\vec{r}',\tau')} = \delta(r-r')\,\delta(\tau-\tau')$

$$-\int dr\, f^{-1}(r-r')\,\bar{n}_i - \langle \vec{r}'\alpha\tau'|\, G_0(\bar{n})\,\sigma^i\,|\vec{r}'\alpha\tau'\rangle = 0$$

$$\int dr\, f^{-1}(r-r') = f^{-1}(q=0)$$

$$n(r) = \frac{1}{V\beta}\sum_{q, i\omega_n} n(q, i\omega_n)\, e^{i(\vec{q}\cdot\vec{r}-\omega_n\tau)}$$

$$f(r-r') = \frac{1}{V}\sum_{\vec{q}} e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} f(q)$$

$$\langle \vec{r}'\alpha\tau' \,|\, G_0(\bar{n})\sigma^i \,|\, \vec{r}'\alpha\tau' \rangle$$

$$= \frac{1}{V\beta}\sum_{\vec{k}, ik_n} tr\left[G_0(k,\vec{\bar{n}})\,\sigma^i \right]$$

$\leftarrow$ small tr refer to spin index

$$\Rightarrow -f^{-1}(q=0)\,\bar{n}_i - \frac{1}{V\beta}\sum_{\vec{k}, ik_n} tr\left[G_0(k,\vec{\bar{n}})\,\sigma^i\right] = 0$$

无妨设 $\bar{n}_i = \bar{n}\,\delta_{i,z} \Rightarrow$

$$\boxed{-f^{-1}(q=0)\,\bar{n} - \frac{1}{V\beta}\sum_{\vec{k}, ik_n} tr\left[G_0(\vec{k}, ik_n; \bar{n})\,\sigma_z\right] = 0}$$

$$G_0(\vec{k}, ik_n; \bar{n}) = \frac{1}{2}\left[\frac{1+\sigma_z}{ik_n-(\varepsilon_k-\mu-\bar{n})} + \frac{1-\sigma_z}{ik_n-(\varepsilon_k-\mu+\bar{n})}\right]$$

$$tr\left[G_0(\vec{k}, ik_n; \bar{n})\,\sigma_z \right] = \frac{1}{ik_n-(\varepsilon_k-\mu-\bar{n})} - \frac{1}{ik_n-(\varepsilon_k-\mu+\bar{n})}$$

因为 $$\frac{1}{\beta}\sum_{ik_n}\frac{1}{ik_n-\varepsilon} = n_f(\varepsilon)$$

$$\Rightarrow \frac{1}{\beta}\sum_{ik_n} tr\left[G_0(\vec{k}, ik_n; \bar{n})\,\sigma_z\right] = n_f(\varepsilon_k - \mu - \bar{n}) - n_f(\varepsilon_k - \mu + \bar{n})$$

$\Rightarrow$ self-consistent Eq

$$-f^{-1}(\vec{q}=0)\,\bar{n} - \frac{1}{V}\sum_k \left[n_f(\varepsilon_{k\uparrow}) - n_f(\varepsilon_{k\downarrow})\right] = 0$$

$$\because f^{-1}(\vec{q}=0) < 0 \Rightarrow \boxed{\frac{\bar{n}}{|f(0)|} = \int\frac{d^3\vec{k}}{(2\pi)^3}\left[n_f(\varepsilon_{k\uparrow}) - n_f(\varepsilon_{\downarrow})\right]}$$

self-consistent Eq for FM.

(2) Gaussian fluctuations

$$\delta S_{eff} = -\frac{1}{2}\int_0^\beta d\tau \int dr\,dr\; f^{-1}(r-r')\,\delta n(r,\tau)\,\delta n(r',\tau)$$

$$+\underbrace{\int d\tau d\tau' \int dr\,dr'}_{\delta n_i(r\tau)\,\delta n_j(r',\tau')} \langle r\alpha\tau|\,G_0(\bar{n})\,\sigma^i\,|r'\beta\tau'\rangle\langle r'\beta\tau'|\,G_0(\bar{n})\,\sigma^i\,|r\alpha\tau\rangle$$

$$\text{or} \int d\tau d\tau' \int dr\,dr'\; \delta n_i(r\tau)\,\delta n_j(r',\tau')\; G_{0,\alpha\alpha}(\bar{n}, r r', \tau\tau')\,\sigma^i_{\alpha\beta}\; G_{0,\beta\beta}\underbrace{(\bar{n}\, r' r, \tau'\tau)}_{\sigma^i_{\beta\alpha}}$$

G_0 is diagonal in spin-index.

$$\boxed{\int d\tau \int dr\,dr'\; f^{-1}(r-r')\,\delta n(r,\tau)\,\delta n(r',\tau) = \frac{1}{V\beta}\sum_{q,i\omega_n} \delta n(q\; i\omega_n)\, f^{-1}(q)\, \delta n(-q,-i\omega_n)}$$

$$= \int d\tau\, dr\,dr'\; \frac{1}{V\beta}\sum_{q\, i\omega_n} e^{iqr-\omega_n\tau}\,\delta n(q,\omega_n)\; \frac{1}{V}\sum_{q'} e^{iq'(r-r')}\, f^{-1}(q)\; \frac{1}{V\beta}\sum_{q'',\omega_n'} e^{iq''x''-\omega_n'\tau}\,\delta n(q'',\omega_n')$$

$$\int d\tau\, d\tau' \int dr\,dr'\; \delta n_i(r,\tau)\,\delta n_j(r',\tau')\; G_{0,\alpha\alpha}(\bar n, r\,r'; \tau\tau')\,\sigma^i_{\alpha\beta}\, G_{0\beta\beta}(\bar n\, r'r, \tau'\tau)\,\sigma^j_{\beta\alpha}$$

plug in $\delta n_i(r,\tau) = \frac{1}{V\beta}\sum_{q, i\omega_n} e^{i(qr-\omega_n\tau)}\,\delta n(q,\omega_n)$

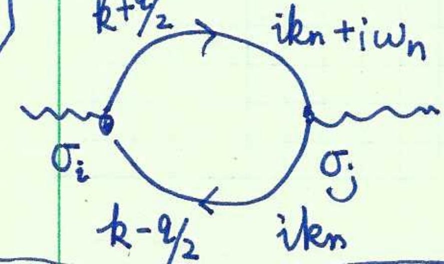
$$G_{0,\alpha\alpha}(\bar n, r r'; \tau\tau') = \frac{1}{V\beta}\, G_{0,\alpha\alpha}(\bar n, k, \omega_n)\, e^{i\left[\vec k(\vec r-\vec r') - \omega_n(\tau-\tau')\right]}$$

$$\Rightarrow \frac{1}{(V\beta)^2}\sum_{\substack{k, ik_n\\ k', ik_n'}} \delta n_i(k-k', ik_n-ik_n')\left[G_{0,\alpha\alpha}(\bar n, k, ik_n)\,\sigma^i_{\alpha\beta}\, G_{0,\beta\beta}(\bar n, k', k_n')\,\sigma^j_{\beta\alpha}\right] \times \delta n_j(k'-k, ik_n'-ik_n)$$

$\Rightarrow$ Gaussian fluctuation

$$\delta S = \frac{1}{2V\beta}\sum_{q, i\omega_n} \delta n_i(q, i\omega_n)\, \mathcal{L}_{ij}(q, i\omega_n)\, \delta n_j(-q,-i\omega_n)$$

where $\mathcal{L}_{ij}(q, i\omega_n) = -f^{-1}(q)\,\delta_{ij} + \frac{1}{V\beta}\sum_{k, i\omega_n} \mathrm{tr}\left[G_0[\bar n, k+q/2, ik_n + i\omega_n]\,\sigma^i\, G_0[\bar n, k-q/2, ik_n]\,\sigma^j \right]$



③ Landau damping in the disordered phase

$$\mathcal{L}_{ij}(q,\omega) = -f^{-1}(q)\,\delta_{ij} + \frac{1}{V\beta}\sum_{k,i\omega_n} \mathrm{tr}[\sigma^i\sigma^j]\,\frac{1}{ik_n+i\omega_n-\varepsilon_{k+q/2}}\,\frac{1}{ik_n-\varepsilon_k}$$

$$= \delta_{ij}\left[-f^{-1}(q) + \frac{2}{V}\sum_k \frac{n_f(\epsilon_{k-q/2}) - n_f(\epsilon_{k+q/2})}{i\omega_n - (\epsilon_{k+q/2}-\epsilon_{k-q/2})}\right]$$

$$= \delta_{ij}\left[-f^{-1}(q) + 2\int\frac{d^3k}{(2\pi)^3}\,\frac{\vec{q}\cdot\hat{k}\,\delta(k-k_f)}{i\omega_n - v_F\hat{k}\cdot\vec{q}}\right]$$

$$= \delta_{ij}\left[-f^{-1}(q) + \frac{2k_f^2}{(2\pi)^3}\int d\Omega\,\frac{q\cos\theta}{i\omega_n - v_F q\cos\theta}\right] \qquad \text{define } s = \frac{\omega}{v_F q}$$

$$i\omega_n \to \omega + i\eta$$

The integral $\rightarrow$ $N_f\int\frac{d\Omega}{4\pi}\left[-1 + \frac{s}{s+i\eta-\cos\theta}\right] = -N_F + s\int\frac{d\Omega}{4\pi}\,\frac{1}{s+i\eta-\cos\theta}$

where $N_f = 2\int\frac{d^3k}{(2\pi)^3}\,\delta[\varepsilon - v_F(k-k_f)] = \frac{2}{v_F}\frac{k_f^2}{(2\pi)^3}\int d\Omega$

$$\int\frac{d\Omega}{4\pi}\,\frac{1}{s+i\eta-\cos\theta} = \frac{1}{2}\int_{-1}^{1} d\cos\theta\,\frac{1}{s+i\eta-\cos\theta} = \frac{-1}{2}\ln\left|\frac{1-s}{1+s}\right| - \frac{\pi}{2}i\,\theta(s<1)$$

$$\Rightarrow \mathcal{L}_{ij}(q,\omega) = \delta_{ij}\left\{-f^{-1}(q) - N_F\left[1 - \frac{s}{2}\ln\left|\frac{1+s}{1-s}\right| + i\frac{\pi}{2}s\,\theta(s<1)\right]\right\}$$

at $s \ll 1$,

$$\mathcal{L}_{ij}(q,\omega) = \delta_{ij}\left\{-f^{-1}(q) - N_F + N_F\left[s^2 - i\frac{\pi}{2}s\right]\right\}$$

$$-f^{-1}(q) = |f_0^{-1}| + \kappa q^2 ,$$ f_0 need to be negative

a paramter put by hand (phenemilogical)

or $f(q) = \dfrac{f_0}{1+\kappa|f_0|q^2}$

or if we keep q^2 order for

define $\delta = |f_0^{-1}| - N_F$, if $\delta > 0 \rightarrow$ paramagnetic phase

$\delta < 0$ — FM phase

$\delta = 0 \Rightarrow \boxed{N_F f_0 = -1} \leftarrow$ Pomenranchuk instability.

onside $\delta > 0 \Rightarrow$

$$\boxed{\mathcal{L}_{ij}(q,\omega) = \delta_{ij}\left(\delta + \kappa q^2 - N_F i\frac{\pi}{2}s\right)}$$

real time

over-damped modes.

$s \propto \frac{-\kappa}{N_F}\frac{2}{\pi} i\, q^2$

$\Rightarrow \boxed{\omega \propto i q^3}$

We also need it's imaginary expression for thermal study later

The key integral is

$$N_F \int \frac{d\Omega}{4\pi} \frac{\frac{i\omega_n}{v_F q}}{\frac{i\omega_n}{v_F q} - \cos\theta} = \frac{N_F}{2}\int_{-1}^{1} dx \frac{\frac{i\omega_n}{v_F q}}{\frac{i\omega_n}{v_F q} - x}$$

$$= \frac{N_F}{2}\cdot\frac{1}{2}\int_{-1}^{1} dx \left[\frac{i\omega_n/v_F q}{i\omega_n/v_F q - x} + \frac{i\omega_n/v_F q}{i\omega_n/v_F q + x}\right]$$

$\frac{\pi}{2} N_F \left|\frac{\omega_n}{v_F q}\right|$ as $\omega_n \ll v_F q$

$$= \frac{N_F}{2}\int_{-1}^{1} dx \frac{\left(\frac{\omega_n}{v_F q}\right)^2}{x^2 + \left(\frac{\omega_n}{v_F q}\right)^2} = \frac{N_F}{2}\left|\frac{\omega_n}{v_F q}\right| \arctan\left(\frac{x}{\left|\frac{\omega_n}{v_F q}\right|}\right)\Bigg|_{-\left|\frac{v_F q}{\omega_n}\right|}^{\left|\frac{v_F q}{\omega_n}\right|}$$

$$\Rightarrow \boxed{\mathcal{L}_{ij}(q, i\omega_n) = \delta_{ij}\left(\delta + xq^2 + \frac{\pi}{2} N_F \left|\frac{\omega_n}{v_F q}\right|\right)}$$

③ RPA calculation of spin-wave spectrum

we need to consider the Gaussian fluctuations in the ordered phase.

Set $\bar{n}_i = \bar{n}\,\delta_{iz}$. $\Rightarrow G_0(k, ik_n) = \frac{1}{2}\left[\frac{1+\sigma_z}{ik_n - \varepsilon_\uparrow(k)} + \frac{1-\sigma_z}{ik_n - \varepsilon_\downarrow(k)}\right]$.

$$\varepsilon_{\uparrow,\downarrow} = \varepsilon_0(k) - \mu \mp \bar{n}$$

Conside transverse modes:

$$A: \frac{1}{V\beta}\sum_{k, i\omega_n} tr\left[G_0[\bar{n}, k+q/2]\,\sigma_x\, G_0[\bar{n}, k-\tfrac{q}{2}]\,\sigma_x\right] \leftarrow \delta n_x(q\omega)\,\delta n_x(-q-\omega)$$

先求 trace $\frac{1}{4} tr\left[\left(\frac{1+\sigma_z}{ik_n + iq_n - \varepsilon_\uparrow(k+q/2)} + \frac{1-\sigma_z}{ik_n + iq_n - \varepsilon_\downarrow(k+q/2)}\right)\sigma_x\left(\frac{1+\sigma_z}{ik_n - \varepsilon_\uparrow(k-q/2)} + \frac{1-\sigma_z}{ik_n - \varepsilon_\downarrow(k-q/2)}\right)\right]$

$$tr[(1\pm\sigma_z)\sigma_x(1\pm\sigma_z)\sigma_x] = tr[\sigma_x\sigma_x + \sigma_z\sigma_x\sigma_z\sigma_x] = 0$$

$$tr[(1\pm\sigma_z)\sigma_x(1\mp\sigma_z)\sigma_x] = tr[\sigma_x\sigma_x - \sigma_z\sigma_x\sigma_z\sigma_x] = 2$$

$\Rightarrow$ The above expression

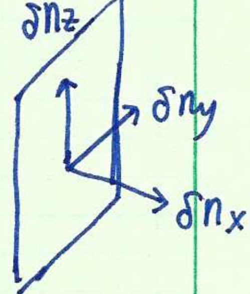
$$\to \frac{1}{2V\beta}\sum_{\vec{k}, ik_n}\left[\frac{1}{ik_n + iq_n - \varepsilon_\uparrow(k+q/2)}\frac{1}{ik_n - \varepsilon_\downarrow(k-q/2)} + \frac{1}{ik_n + iq_n - \varepsilon_\downarrow(k+q/2)}\frac{1}{ik_n - \varepsilon_\uparrow(k-q/2)}\right]$$

define $A(q, iq_n) = \frac{1}{V\beta}\sum_{k, ik_n}\frac{1}{ik_n + iq_n - \varepsilon_\uparrow(k+q/2)}\frac{1}{ik_n - \varepsilon_\downarrow(k-q/2)}$

$$B(q, iq_n) = \frac{1}{V\beta}\sum_{k, ik_n}\frac{1}{ik_n + iq_n - \varepsilon_\downarrow(k+q/2)}\frac{1}{ik_n - \varepsilon_\uparrow(k-q/2)}$$

$\Leftrightarrow \boxed{A(-q, -iq_n) = B(q, iq_n)}$

$\delta n_x, \delta n_y$ 耦合



if $\delta n_z \neq 0$, no verticle reflection plane sym.

$$\Rightarrow \boxed{\frac{1}{2}\delta n_x(q,\omega)\,\delta n_x(-q,-\omega)\left[A(q\,\omega)+B(q,\omega)\right]}$$

Similarly for y-y

$$\boxed{\frac{1}{2}\delta n_y(q\,\omega)\,\delta n_y(-q,-\omega)\left[A(q\,\omega)+B(q\,\omega)\right]}$$

consider $\delta n_x(q\,\omega)\,\delta n_y(-q-\omega)$

$$tr\left[G(\vec{n},k+q/2)\,\sigma_x\,G(\vec{n},k-q/2)\,\sigma_y\right]$$

$$= \frac{1}{4}tr\left[\left(\frac{1+\sigma_z}{ik_n+iq_n-\varepsilon_\uparrow(k+q/2)}+\frac{1-\sigma_z}{ik_n+iq_n-\varepsilon_\downarrow(k+q/2)}\right)\sigma_x\left(\frac{1+\sigma_z}{ik_n-\varepsilon_\uparrow(k-q/2)}+\frac{1-\sigma_z}{ik_n-\varepsilon_\downarrow(k-q/2)}\right)\sigma_y\right]$$

$$tr\left[(1\pm\sigma_z)\,\sigma_x\,(1\pm\sigma_z)\,\sigma_y\right] = \pm tr\left[\sigma_z\sigma_x\sigma_y\right] \pm tr\left[\sigma_x\sigma_z\sigma_y\right] = 0$$

$$tr\left[(1\pm\sigma_z)\,\sigma_x\,(1\mp\sigma_z)\,\sigma_y\right] = \pm tr\left[\sigma_z\sigma_x\sigma_y\right] \mp tr\left[\sigma_x\sigma_z\sigma_y\right] = \pm 2i$$

$$\rightarrow \boxed{\frac{i}{2}\delta n_x(q\,\omega)\,\delta n_y(-q,-\omega)\left[A(q,\omega)-B(q,\omega)\right]}$$

similarly

$$\boxed{\frac{-i}{2}\delta n_y(q\,\omega)\,\delta n_x(-q,-\omega)\left[A(q\,\omega)-B(q\,\omega)\right]}$$

$$\Rightarrow \mathcal{L} = \sum_{q\omega}\left\{\frac{1}{\sqrt{2}}\left[\delta n_x(q,\omega)+i\delta n_y(q\omega)\right]\right\}\left\{\frac{\delta n_x(-q-\omega)-i\delta n_y(-q,-\omega)}{\sqrt{2}}\right\} B(q,\omega)$$

$$+\left\{\frac{\delta n_x(q\omega)-i\delta n_y(q\omega)}{\sqrt{2}}\right\}\left\{\frac{\delta n_x(-q-\omega)+i\delta n_y(-q,-\omega)}{\sqrt{2}}\right\} A(q,\omega)$$

不独立，实为一个表达式。

$$= \sum_{q,\omega}\left(\delta n_x(q\,\omega)+i\delta n_y(q,\omega)\right)\left(\delta n_x(-q-\omega)-i\delta n_y(-q,-\omega)\right) B(q,\omega)$$

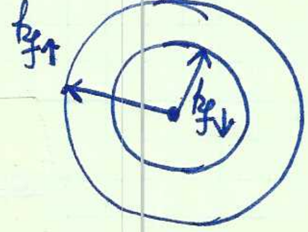
define $\delta n_\pm(q,\omega) = \delta n_x(q\,\omega) \pm i\,\delta n_y(q,\omega)$

$$\mathcal{L} = \sum_{q\omega} \delta n_+(\vec{q}\,\omega)\,\delta n_-(-\vec{q},-\omega)\left[\frac{\chi}{2}q^2 + \frac{1}{2|f_0|} + B(q,\omega)\right]$$

$\delta n_\pm$ are complex field, $\left[n_+(\vec{q},\omega)\right]^* = n_-(-\vec{q},-\omega)$.

$$B(q,\omega) = \frac{1}{V\beta}\sum_{k,ik_n} \frac{1}{ik_n+\omega-\varepsilon_\downarrow(k+q/2)}\,\frac{1}{ik_n-\varepsilon_\uparrow(k-q/2)}$$

$$= \frac{1}{V}\sum_k \frac{n_f(\varepsilon_\uparrow(k-q/2)) - n_f(\varepsilon_\downarrow(k+q/2))}{\omega+\varepsilon_\uparrow(k-q/2)-\varepsilon_\downarrow(k+q/2)}$$



$$\varepsilon_{\uparrow,\downarrow}(k-q/2) = \varepsilon(k-q/2) \mp n$$

$$\Rightarrow B(q\,\omega) = \frac{1}{V}\sum_k\left[\frac{n_f(\varepsilon_\uparrow(k))}{\omega-2\bar{n}+\varepsilon(k)-\varepsilon(k+q)} - \frac{n_f(\varepsilon_\downarrow(k))}{\omega-2\bar{n}+\varepsilon(k-q)-\varepsilon(k)}\right]$$

$$\varepsilon_{\vec{k}\pm\vec{q}} - \varepsilon_{\vec{k}} = \pm\vec{q}\cdot\vec{\nabla}\,\varepsilon_k + \frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k$$

$$\frac{1}{\omega-2\bar{n}-(\vec{q}\cdot\vec{\nabla})\varepsilon_k-\frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k} = -\frac{1}{2n}\left[\frac{1}{1-\frac{\omega-(\vec{q}\cdot\nabla)\varepsilon_k-\frac{1}{2}(\vec{q}\cdot\nabla)^2\varepsilon_k}{2n}}\right]$$

$$= -\frac{1}{2n}\left\{1 + \frac{\omega-(\vec{q}\cdot\nabla)\varepsilon_k\cancel{-\frac{1}{2}(\vec{q}\cdot\nabla)^2\varepsilon_k}}{2n} + \frac{\left[\omega-(\vec{q}\cdot\nabla)\varepsilon_k-\frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k\right]^2}{4n^2}\right.$$

$$\sim \frac{-1}{2n}\left\{1 + \frac{\omega-\frac{1}{2}(\vec{q}\cdot\nabla)^2\varepsilon_k}{2n} + \frac{\omega^2+\left[(\vec{q}\cdot\nabla)\varepsilon_k\right]^2}{4n^2}\right\}$$ ← 含 $\vec{q}\cdot\nabla$ 奇次项都消去了，在对角度积分后

$\frac{\omega^2}{4n^2}$ 可以略去，因为 $\omega\sim q^2$，这个 term 也是高阶的在色散关系处.

$$\Rightarrow \frac{1}{\omega - 2\bar{n} - (\vec{q}\cdot\vec{\nabla})\varepsilon_k - \frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k} \sim \frac{-1}{2n}\left[1 + \frac{\omega}{2n} - \frac{(q\cdot\nabla)^2\varepsilon_k}{4n} + \frac{(q\cdot\nabla\varepsilon_k)^2}{4n^2}\right]$$

Similarly

$$\frac{1}{\omega - 2\bar{n} + \varepsilon(k-q) - \varepsilon(k)} = \frac{1}{\omega - 2\bar{n} - (\vec{q}\cdot\vec{\nabla})\varepsilon_k + \frac{1}{2}(\vec{q}\cdot\nabla)^2\varepsilon_k}$$

$$\sim \frac{-1}{2n}\left[1 + \frac{\omega}{2n} + \frac{(q\cdot\nabla)^2\varepsilon_k}{4n} + \frac{(q\cdot\nabla\varepsilon_k)^2}{4n^2}\right]$$

$$\Rightarrow B(q\,\omega) = \int_{k_{f\downarrow}}^{k_{f\uparrow}} \frac{d^3k}{(2\pi)^3}\left(\frac{-1}{2\bar{n}}\right) + \int_{k_{f\downarrow}}^{k_{f\uparrow}} \frac{d^3k}{(2\pi)^3}\frac{-\omega}{(2\bar{n})^2} - \underbrace{\int_{k_{f\downarrow}}^{k_{f\uparrow}} \frac{d^3k}{(2\pi)^3}\frac{(q\cdot\nabla\varepsilon_k)^2}{8n^3} + \int \frac{d^3k}{(2\pi)^3}\frac{(n_f(\varepsilon_\uparrow)+n_f(\varepsilon_\downarrow))(q\cdot\nabla)^2\varepsilon_k}{8n^2}}_{\text{give correction to spin-wave stiffness.}}$$

Due to self-consistent Eq.
$$\boxed{\frac{1}{|f_0|} = \int_{k_{f\downarrow}}^{k_{f\uparrow}} \frac{d^3k}{(2\pi)^3}\frac{1}{\bar{n}}}$$

$$\Rightarrow \frac{\chi}{2}q^2 + \frac{1}{2|f_0|} + B(q,\omega) = \frac{-\omega}{4\bar{n}}\frac{1}{|f_0|} + \frac{\chi'}{2}q^2$$

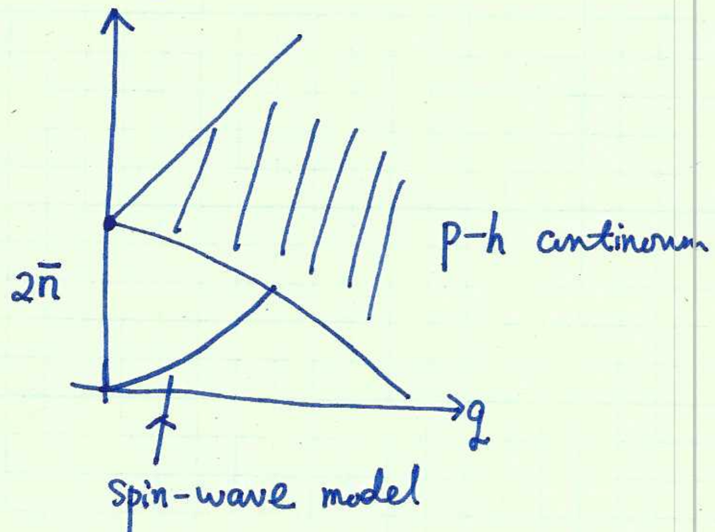
where
$$\chi' = \chi + \frac{1}{8n^2}\int \frac{d^3k}{(2\pi)^2}\left\{\frac{[n_f(\varepsilon_\uparrow)+n_f(\varepsilon_\downarrow)]}{m} - \frac{(n_f(\varepsilon_\uparrow)-n_f(\varepsilon_\downarrow))\,v_F}{n}\right\}$$

$$\Rightarrow \boxed{\delta\mathcal{L} = \sum_{q\omega} \delta n_+(\vec{q},\omega)\,\delta n_-(\vec{q},\omega)\left[-\frac{\omega}{4\bar{n}}\frac{1}{|f_0|} + \frac{\chi'}{2}q^2\right]}$$

spin-wave

linear in $\omega \Rightarrow \delta n_+$ and δn_- are conjugate to each other.

$$\omega_{kq} = \varepsilon_{k+q,\downarrow} - \varepsilon_{k\uparrow} = 2n + \varepsilon_{k+q} - \varepsilon_k$$



outside the p-h continuum.