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Supplementary material: an introduction to second quantization

§1 Harmonic oscillator (warm up)

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2, \quad \text{define characteristic length } l = \sqrt{\frac{\hbar}{m\omega}}$$

then  $a = \frac{1}{\sqrt{2}} \left[ \frac{x}{l} + i p l \right], \quad a^\dagger = \frac{1}{\sqrt{2}} \left[ \frac{x}{l} - i p l \right]$

$$[a, a^\dagger] = 1, \quad \text{and} \quad H = \hbar \omega (a^\dagger a + \frac{1}{2}).$$

The eigenstate of oscillator can be expressed as  $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$

where  $|0\rangle$  satisfies  $a|0\rangle = 0$ .

ex: ①  $\langle x|0\rangle = \frac{1}{\sqrt{\pi^{1/2} l}} e^{-\frac{x^2}{2l^2}}$

②  $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad a^\dagger a |n\rangle = n |n\rangle$

§2 Bose statistics and Fermi statistics

suppose we have a set of complete and normalized basis of single particle wavefunctions, we can use them to construct the  $N$ -body wavefunction.

$\psi_1, \psi_2, \dots$

For bosons, we first consider

$$\underbrace{\psi_1(x_1) \dots \psi_1(x_{n_1})}_{N_1} \psi_2(x_{n_1+1}) \dots \psi_2(x_{n_1+n_2}) \dots \underbrace{\{\psi_k(x_1) \dots \psi_k(x_N)\}}_{N_k}$$

and symmetrize it

$$\bar{\Psi}_{N_1 \dots N_k}(\xi_1 \dots \xi_N) = \left( \frac{N!}{N_1! \dots N_k!} \right)^{-1/2} \sum_{P \in \mathcal{P}_E} \underbrace{\psi_1(x_1) \dots \psi_1(x_{N_1})}_{N_1} \dots \underbrace{\psi_k(x_{N_1+\dots+N_{k-1}+1}) \dots \psi_k(x_N)}_{N_k}$$

For fermions, each state can only support one particle

$$\psi_{i_1}(\xi_1) \psi_{i_2}(\xi_2) \cdots \psi_{i_N}(\xi_N)$$

$$\rightarrow \Psi_{\{i_1, \dots, i_N\}}(\xi_1, \dots, \xi_N) = \frac{1}{\sqrt{N!}} \sum_P (-)^P P \psi_{i_1}(\xi_1) \psi_{i_2}(\xi_2) \cdots \psi_{i_N}(\xi_N)$$

### §3: Second quantization for bosons

From the above section, we learn that as long as we have a set of complete and orthogonal single particle basis, and specify the occupation number distribution, we can write down the many body wavefunction. This set of

many body wavefunction basis is basis for characterize by the particle numbers in the states of  $\psi_1, \psi_2, \dots$  as  $N_1, N_2, \dots$ . We define the Fock space as formed

by all the eigenstates of particle number operator  $\hat{n}_1, \hat{n}_2, \dots$ . The transformation

between different basis is through the creation/annihilation operator of each single particle state.

$$\text{Basis: } \Psi_{N_1, N_2, \dots}(\xi_1, \xi_2, \dots, \xi_N) = \sqrt{\frac{N_1! \cdots N_k! \cdots}{N!}} \sum_P P \underbrace{\{\psi_1(\xi_1) \cdots \psi_1(\xi_{N_1})\}}_{N_1} \underbrace{\{\psi_2(\xi_{N_1+1}) \cdots\}}_{N_2} \cdots$$

N-body wavefunction  $\Psi$  can be expanded as

$$\Psi(\xi_1, \xi_2, \dots, \xi_N) = \sum_{N_1, N_2, \dots} \Psi_{N_1, N_2, \dots}(\xi_1, \xi_2, \dots, \xi_N) C(N_1, N_2, \dots)$$

$$\text{where } C(N_1, N_2, \dots) = \begin{cases} 0 & \text{if } \sum_i N_i \neq N \\ (\Psi_{N_1, N_2, \dots}, \Psi) & \text{if } \sum_i N_i = N. \end{cases}$$

we can define inner product of two wave functions as

$$(\Psi_A, \Psi_B) = \sum_{N_1, N_2, \dots} C_A^*(N_1, N_2, \dots) C_B(N_1, N_2, \dots).$$

Thus all the theory can be represented by using the particle number representation in which the wavefunction is written as  $C(N_1, N_2, \dots)$ . The many-particle wavefunction basis defined above  $\Psi_{N'_1, N'_2, \dots, N'_k}$  in this representation is

$$C_{\underbrace{N'_1, N'_2, \dots, N'_k}_{\text{indices of basis}}}(N_1, N_2, \dots) = \delta_{N_1, N'_1} \delta_{N_2, N'_2} \dots$$

↑ arguments of variables

More conveniently, we define the ket-space as

$$C_{N'_1, N'_2, \dots, N'_k}(N_1, N_2, \dots) \longleftrightarrow |N'_1, N'_2, \dots\rangle$$

$$\alpha C_{N'_1, N'_2, \dots}(N_1, N_2) + \beta C_{N''_1, N''_2, \dots}(N_1, N_2) \longleftrightarrow \alpha |N'_1, N'_2, \dots\rangle + \beta |N''_1, N''_2, \dots\rangle$$

$$\langle N'_1, N'_2, \dots | N''_1, N''_2, \dots \rangle = \sum_{N_1, N_2} (\delta_{N_1, N'_1} \delta_{N_2, N'_2} \dots) (\delta_{N_1, N''_1} \delta_{N_2, N''_2} \dots)$$

$$= \delta_{N'_1, N''_1} \delta_{N'_2, N''_2} \dots$$

and  $\sum_{N_1, N_2, \dots} |N_1, N_2, \dots\rangle \langle N_1, N_2, \dots| = 1$

and  $C(N_1, N_2, \dots) = \langle N_1, N_2, \dots | \psi \rangle$ , where  $\psi$  is an arbitrary state vector,

### ★ Creation/annihilation operators

particle number operators  $\hat{n}_i |N_1, N_2, \dots\rangle = N_i |N_1, N_2, \dots\rangle$

$$\hat{n}_i^\dagger = n_i, \quad [\hat{n}_i, \hat{n}_j] = 0$$

we define operators

$$a_i = \sum_{N_1, N_2, \dots} \sqrt{N_i} |N_1, N_2, \dots, (N_i-1), \dots\rangle \langle N_1, N_2, \dots, N_i, \dots|,$$

$$a_i^\dagger = \sum_{N_1, N_2, \dots} \sqrt{N_i+1} |N_1, N_2, \dots, (N_i+1), \dots\rangle \langle N_1, N_2, \dots, N_i, \dots|,$$

$$\Rightarrow a_i |N_1, N_2, \dots, N_i, \dots\rangle = \sqrt{N_i} |N_1, N_2, \dots, (N_i-1), \dots\rangle$$

$$a_i^\dagger |N_1, N_2, \dots, N_i, \dots\rangle = \sqrt{N_i+1} |N_1, N_2, \dots, (N_i+1), \dots\rangle$$

check  $[a_i, a_j^\dagger] = \delta_{ij}$ ,  $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$

define that the vacuum  $|0\rangle = |0, 0, \dots, 0, \dots\rangle \Rightarrow$

$$|N_1, N_2, \dots\rangle = \frac{(a_1^\dagger)^{N_1}}{\sqrt{N_1!}} \frac{(a_2^\dagger)^{N_2}}{\sqrt{N_2!}} \dots |0\rangle$$

### ★ Field operator

$$\hat{\psi}(r) = \sum_i \psi_i(r) a_i$$

$$\hat{\psi}^\dagger(r) = \sum_i \psi_i^*(r) a_i^\dagger$$

single particle wavefunction

a parameter

Let's check the physical meaning of  $\hat{\psi}^\dagger(r)$ , Define  $|r'\rangle$  the single particle position eigenstate

$$\langle r' | \hat{\psi}^\dagger(r) | 0 \rangle = \sum_i \langle r' | a_i^\dagger | 0 \rangle \psi_i^*(r)$$

$$= \sum_i \psi_i(r) \psi_i^*(r') = \delta(r'-r) \text{ which doesn't depend on the}$$

basis.



thus  $\hat{\psi}^\dagger(r)$  means the creation of one particle at the location  $r$ ,  
it's the creation operator in the coordinate representation.

$\hat{\psi}^\dagger(r)\hat{\psi}(r)$  is the density operator at location  $r$ , and

$$[\hat{\psi}(r)\hat{\psi}^\dagger(r')] = \sum_{ij} \psi_i(r)\psi_j^\dagger(r') [a_i, a_j^\dagger] = \sum_i \psi_i(r)\psi_i^\dagger(r') = \delta(r-r')$$

$$[\psi(r)\psi(r')] = [\psi^\dagger(r), \psi^\dagger(r')] = 0$$

★ transformation of creation/annihilation operators in different basis.

suppose that we define a set of operators  $a_i, a_i^\dagger$  associated with basis  $\psi_i$ , and a set of operators  $b_i, b_i^\dagger$  associated with basis of  $\phi_i$ .

Because the field operator is independent of basis, i.e.

$$\hat{\psi}(r) = \sum_i \psi_i(r) a_i = \sum_i \phi_i(r) b_i$$

$$\Rightarrow a_i = \sum_j \langle \psi_i | \phi_j \rangle b_j, \quad a_i^\dagger = \sum_j \langle \phi_j | \psi_i \rangle b_j^\dagger$$

for example  $\hat{\psi}(r) = \sum_k \frac{1}{\sqrt{V}} e^{ikr} a_k$ ,

$$a_k = \int d^3r \frac{e^{-ikr}}{\sqrt{V}} \hat{\psi}(r)$$

★ expression of operators / Schrödinger equation

1) single body operator

$F = \sum_{p=1}^N f(p)$  in the first quantization, where  $f(p)$  only depends on the variable of the  $p$ -th particle.

let us begin with a representation in which  $f$  is diagonal,

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$f \psi_k = f_k \psi_k$ , then

$$\hat{F} |N_1 N_2 \dots\rangle = (N_1 f_1 + N_2 f_2 + \dots) |N_1 N_2 \dots\rangle = \sum_k f_k \hat{n}_k |N_1 N_2 \dots\rangle$$

$$\Rightarrow \hat{F} = \sum_k f_k a_k^\dagger a_k = \sum_k \langle k | f | k \rangle a_k^\dagger a_k \quad \text{in the diagonal basis.}$$

let us change to a general basis  $|\phi_i\rangle$  with associated operators  $b_i^\dagger$

$$\Rightarrow a_k = \sum_i \langle k | \phi_i \rangle b_i, \quad a_k^\dagger = \sum_i \langle \phi_i | k \rangle b_i^\dagger$$

$$\Rightarrow \hat{F} = \sum_{k, i, j} \langle \phi_j | k \rangle \langle k | f | k \rangle \langle k | \phi_i \rangle b_j^\dagger b_i = \sum_{ij} \langle \phi_j | f | \phi_i \rangle b_j^\dagger b_i$$

thus for a general basis

$$\hat{F} = \sum_{ij} \langle i | f | j \rangle b_i^\dagger b_j$$

example: in the coordinate basis

$$\hat{F} = \int dr dr' \langle r | f | r' \rangle \hat{\psi}^\dagger(r) \hat{\psi}(r')$$

if the coordinate rep.  $\hat{f} = f(r, \nabla_r)$

$$\langle r | f | r' \rangle = \int dx \delta(x-r) f(x, \nabla_x) \delta(x-r') = f(r, \nabla_r) \delta(r-r')$$

$$\hat{F} = \int dr dr' \{ f(r, \nabla_r) \delta(r-r') \} \hat{\psi}^\dagger(r) \hat{\psi}(r') = \int dr \hat{\psi}^\dagger(r) f(r, \nabla_r) \hat{\psi}(r)$$

$$H_0 = \sum_k \frac{\hbar^2 k^2}{2m} a_k^\dagger a_k \quad (\text{plane wave})$$

$$H = \sum_{\langle ij \rangle} c_i^\dagger c_j \langle i | h | j \rangle \quad (\text{tight-binding model})$$



generally, speaking.  $\hat{G}$  can be expanded into a set of  $U$ s and  $V$ s.

the above expression still valid,  $\Rightarrow$

$$\hat{G} = \frac{1}{2} \sum_{i l k m} \langle i k | g | l m \rangle a_i^\dagger a_k^\dagger a_m a_l$$

$$\langle i k | g | l m \rangle = \int dr_1 dr_2 \psi_i^*(1) \psi_k^*(2) g(1,2) \psi_m(2) \psi_l(1)$$

or in the coordinate Rep: in terms of field operator

$$\hat{G} = \frac{1}{2} \int dr_1 dr_2 \hat{\psi}^\dagger(r_1) \hat{\psi}^\dagger(r_2) g(r_1, r_2) \psi(r_2) \psi(r_1)$$

$\rightarrow$  momentum space if  $g(r_1, r_2) = g(r_1 - r_2)$ , we have

$$\hat{G} = \frac{1}{2V} \sum_{k_1 k_2 q} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_2 - q} a_{k_1 + q} g(q), \quad g(q) = \int dr e^{-i q r} g(r)$$

$\Rightarrow$  The many-body Hamiltonian  $\Rightarrow$

$$H = \int dr \hat{\psi}^\dagger(r) [T + U] \psi(r) + \frac{1}{2} \int dr_1 dr_2 \hat{\psi}^\dagger(r_1) \hat{\psi}^\dagger(r_2) (V(1,2)) \psi(r_2) \psi(r_1)$$

quantization of  $\psi \rightarrow$  second quantization.

or  $H = \sum_{i i'} \langle i | T + U | i' \rangle a_i^\dagger a_{i'} + \frac{1}{2} \sum_{\substack{i k \\ i' k'}} \langle i k | V | i' k' \rangle a_i^\dagger a_k^\dagger a_{k'} a_{i'}$

tight-binding

$$\langle i k | V | i' k' \rangle = \int dr_1 dr_2 \phi_i(1) \phi_k(2) V(1,2) \phi_{k'}(2) \phi_{i'}(1)$$



### §4. Second quantization for fermions

Again we define the many-body wavefunction basis ~~with~~ with occupation numbers. We give the sequence  $\{\psi_i\}$   $i=1,2,3,\dots$ , with particle number distribution  $N_1, N_2, N_3, \dots$ ,  $N_i$  can only be 0 or 1.

$\downarrow \swarrow$   
 index of single particle wavefunctions

Pauli-exclusion / Fermi statistics, exchange one pair of particle  $\Psi \rightarrow -\Psi$

$$\begin{aligned} \Psi_{N_1 N_2 N_3 \dots}(\xi_1 \dots \xi_N) &= (-)^{N_e N_{e-1}} \Psi_{N_1 N_2 \dots N_e N_{e-1} N_{e+1} \dots}(\xi_1 \dots \xi_N) \\ &= (-)^{N_e \sum_{j=1}^{e-1} N_j} \Psi_{N_e N_1 N_2 \dots}(\xi_1 \dots \xi_N) \quad (\text{many } N\text{-body basis}) \end{aligned}$$

Again we can expand any ~~many~~ N-body wavefunction

$$\Psi(\xi_1 \dots \xi_N) = \sum_{N_1 N_2 \dots} \Psi_{N_1 N_2 \dots}(\xi_1 \dots \xi_N) C(N_1 N_2 \dots)$$

wavefunction in the particle number Rep

Again we introduce Ket-vector

$$\begin{aligned} C_{N_1' N_2' \dots}(N_1, N_2, \dots) &\leftrightarrow |N_1' N_2' \dots\rangle \\ \underbrace{\hspace{1.5cm}}_{\text{index of basis}} \quad \underbrace{\hspace{1.5cm}}_{\text{variable}} \end{aligned}$$

$$\alpha C_{N_1' N_2'}(N_1, N_2, \dots) + \beta C_{N_1'' N_2''}(N_1, N_2, \dots) \leftrightarrow \alpha |N_1' N_2' \dots\rangle + \beta |N_1'' N_2'' \dots\rangle$$

$$\begin{aligned} |N_1 N_2 \dots N_{e-1} N_e \dots\rangle &= (-)^{N_e N_{e-1}} |N_1 N_2 \dots N_e N_{e-1} \dots\rangle \\ &= (-)^{N_e \sum_{j=1}^{e-1} N_j} |N_e N_1 N_2 \dots\rangle \end{aligned}$$

$$\sum_{N_1 N_2 \dots} |N_1 N_2 \dots\rangle \langle N_1 N_2 \dots| = \mathbb{1}, \quad C(N_1 N_2 \dots) = \langle N_1 N_2 \dots | \Psi \rangle$$

Again, we define particle number operator

$$\hat{n}_i |N_1 \dots N_i \dots\rangle = N_i |N_1 \dots N_i \dots\rangle$$

and 
$$\hat{n}_i = \sum_{N_1, N_2, \dots} N_i |N_1, N_2, \dots\rangle \langle N_1, N_2, \dots|,$$

$$\hat{n}_i^\dagger = n_i, \quad [\hat{n}_i, \hat{n}_{i'}] = 0$$

annihilation / creation = 
$$a_i^\dagger |N_1 \dots N_{i-1} 0_i N_{i+1} \dots\rangle = (-)^{\sum_{l=1}^{i-1} N_l} |N_1 \dots N_2 1_i N_{i+1} \dots\rangle$$

$$a_i^\dagger |N_1 \dots 1_i \dots\rangle = 0$$

$$\Rightarrow a_i^\dagger = \sum_{N_1, N_2, \dots} (-)^{\sum_{l=1}^{i-1} N_l} |N_1, N_2, \dots, 1_i, \dots\rangle \langle N_1, N_2, \dots, 0_i, \dots|$$

$$a_i = \sum_{N_1, N_2, \dots} (-)^{\sum_{l=1}^{i-1} N_l} |N_1, N_2, \dots, 0_i, \dots\rangle \langle N_1, N_2, \dots, 1_i, \dots|$$

Then  $\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_i\} = \{a_i^\dagger, a_i^\dagger\} = 0$

$$|N_1, N_2, \dots\rangle = (a_1^\dagger)^{N_1} (a_2^\dagger)^{N_2} \dots |0\rangle$$

$$|N_2, N_1, \dots\rangle = (a_2^\dagger)^{N_2} (a_1^\dagger)^{N_1} \dots |0\rangle = (-)^{N_1 N_2} |N_1, N_2, \dots\rangle$$

again we define field operators which are annihilation/creation operators

in the coordinate Rep

$$\hat{\psi}(r) = \sum_i \psi(r) a_i; \quad \hat{\psi}^\dagger(r) = \sum_i \psi_i^*(r) a_i^\dagger$$

which satisfy  $\{\hat{\psi}(r), \hat{\psi}^\dagger(r')\} = \delta(r-r'), \quad \{\hat{\psi}(r), \hat{\psi}(r')\} = 0$

\* operators represented by second quantization

$$F = \sum_{p=1}^N f(p) \rightarrow \hat{F} = \sum_{ik} \langle il | f | k \rangle a_i^\dagger a_k = \int dr \hat{\psi}^\dagger(r) f \psi(r)$$

$$G = \sum_{p < q}^N g(p, q) \rightarrow \hat{G} = \frac{1}{2} \sum_{iklm} \langle ik | g | lm \rangle a_i^\dagger a_k^\dagger a_m a_l$$

where  $\langle i k | g | l m \rangle = \int d\xi_1 d\xi_2 \psi_i^*(\xi_1) \psi_j^*(\xi_2) g(1,2) \psi_l(\xi_1) \psi_m(\xi_2)$

$\rightarrow \hat{G} = \int d r_1 d r_2 \psi^\dagger(r_1) \psi^\dagger(r_2) g(1,2) \psi(r_2) \psi(r_1)$

if we have spin  $\Rightarrow \hat{G} = \int d r_1 d r_2 \psi_{\sigma}^\dagger(r_1) \psi_{\sigma'}^\dagger(r_2) g(1,2) \psi_{\sigma'}(r_2) \psi_{\sigma}(r_1)$

The many body hamiltonian

$H = \int d r \hat{\psi}_{\sigma}^\dagger(r) (T + U) \hat{\psi}_{\sigma}(r) + \frac{1}{2} \int d r_1 d r_2 \hat{\psi}_{\sigma}^\dagger(r_1) \hat{\psi}_{\sigma'}^\dagger(r_2) V(1,2) \hat{\psi}_{\sigma'}(r_2) \hat{\psi}_{\sigma}(r_1)$

$= \sum_{i i'} \langle i | T + U | i' \rangle a_i^\dagger a_{i'} + \frac{1}{2} \sum_{i k i' k'} \langle i k | V | i' k' \rangle a_i^\dagger a_k^\dagger a_{k'} a_{i'}$

§ Examples :

- ① Evaluation of the Hartree - Fock interaction energy of the state  $|G\rangle$  in which the  $|k\sigma\rangle$  inside the Fermi sphere ( $k_F$ ) is occupied.

$V = \frac{1}{2V} \sum_{k_1 k_2 q} V(q) a_{k_1+q\sigma}^\dagger a_{k_2-q\sigma'}^\dagger a_{k_2\sigma'} a_{k_1\sigma}$

$|G\rangle = \prod_{k < k_F} a_{k\uparrow}^\dagger a_{k\downarrow}^\dagger |0\rangle$

We need to evaluate  $\langle G | V | G \rangle$ .

Hartree term:

$g=0 \quad V_H = \frac{1}{2V} \sum_{k_1 k_2} V(0) \langle G | a_{k_1\sigma}^\dagger a_{k_2\sigma'}^\dagger a_{k_2\sigma'} a_{k_1\sigma} | G \rangle$

$$= \frac{1}{2V} \sum_{k_1, k_2} V(0) \langle G | a_{k_1, \sigma}^\dagger a_{k_1, \sigma} a_{k_2, \sigma'}^\dagger a_{k_2, \sigma'} - a_{k_1, \sigma}^\dagger a_{k_2, \sigma'} \delta_{k_1, k_2} \delta_{\sigma, \sigma'} | G \rangle$$

$$= \frac{V(0)}{2V} \left[ \left( \sum_{k, \sigma} n_{k, \sigma} \right)^2 - \sum_k (n_{k, \uparrow} + n_{k, \downarrow}) \right] = \frac{1}{2} V(0) [N^2 - n]$$

(This energy in the interacting electron system can be canceled by the positive background charge).

Fock term:  $\sigma = \sigma', \& k_2 - q = k_1, \text{ and } k_1 \neq k_2$

$$V_{\text{Fock}} = \frac{1}{2V} \sum_{k_1 \neq k_2} V(k_1 - k_2) \langle G | a_{k_2, \sigma}^\dagger a_{k_1, \sigma}^\dagger a_{k_2, \sigma} a_{k_1, \sigma} | G \rangle$$

$$= \frac{-1}{2V} \sum_{k_1, k_2} V(k_1 - k_2) \langle G | a_{k_2, \sigma}^\dagger a_{k_2, \sigma'}^\dagger a_{k_1, \sigma}^\dagger a_{k_1, \sigma'} | G \rangle = \frac{-1}{2V} \sum_{k_1, k_2} V(k_1 - k_2) n_{k_2} n_{k_1} \cdot 2$$

$$= -\frac{1}{V} \cdot V^2 \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} V(k_1 - k_2) n_{k_1} n_{k_2} = -V_0 \int \frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^3} n_{k_1} n_{k_2} V(k_1 - k_2)$$

② Cooper pair problem!

Consider that we have a full-filled Fermi sphere with Fermi wavevector  $k_F$ . We add two electrons with  $(k \uparrow)$  and  $(-k \downarrow)$ . Neglect that electrons inside the Fermi sphere can be scattered outside the Fermi surface. Assume attractive interaction between these two electrons.

$$H = \sum_k \epsilon_k C_{k, \sigma}^\dagger C_{k, \sigma} - V \sum_{k, k'} C_{k, \uparrow}^\dagger C_{-k, \downarrow}^\dagger C_{-k, \downarrow} C_{k, \uparrow}$$

Solve the spectrum for these two electrons.



interactions cause scattering  $(k \uparrow; -k \downarrow) \rightarrow (k' \uparrow; -k' \downarrow)$ .

the eigenstate should be a linear superposition of these states.

$$|\psi\rangle = \sum_{\mathbf{k}} \alpha(\mathbf{k}) C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle, \quad \text{where } \alpha(\mathbf{k}) \text{ is the coefficient}$$

$|F\rangle$  the Full-filled Fermi sphere.

$$H|\psi\rangle = \sum_{\mathbf{k}} \alpha(\mathbf{k}) (H_0 C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} + U) |F\rangle$$

$$H_0 C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle = (C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} H_0 |F\rangle + 2E_{\mathbf{k}}) C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle = (2E_{\mathbf{k}} + E_0) C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle$$

$$U C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle = U \sum_{\mathbf{k}'\mathbf{k}''} C_{\mathbf{k}'\uparrow}^{\dagger} C_{-\mathbf{k}'\downarrow}^{\dagger} C_{-\mathbf{k}''\downarrow} C_{\mathbf{k}''\uparrow} C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle$$

$$= U \sum_{\mathbf{k}'\mathbf{k}''} \delta(\mathbf{k}, \mathbf{k}'') C_{\mathbf{k}'\uparrow}^{\dagger} C_{-\mathbf{k}'\downarrow}^{\dagger} |F\rangle = U \sum_{\mathbf{k}'} C_{\mathbf{k}'\uparrow}^{\dagger} C_{-\mathbf{k}'\downarrow}^{\dagger} |F\rangle$$

$$\Rightarrow H|\psi\rangle = \sum_{\mathbf{k}} \alpha(\mathbf{k}) [2E_{\mathbf{k}} + E_0] C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle + \sum_{\mathbf{k}} \underbrace{\alpha(\mathbf{k})}_U \sum_{\mathbf{k}'} C_{\mathbf{k}'\uparrow}^{\dagger} C_{-\mathbf{k}'\downarrow}^{\dagger} |F\rangle$$

$$= E \sum_{\mathbf{k}} \alpha(\mathbf{k}) C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle$$

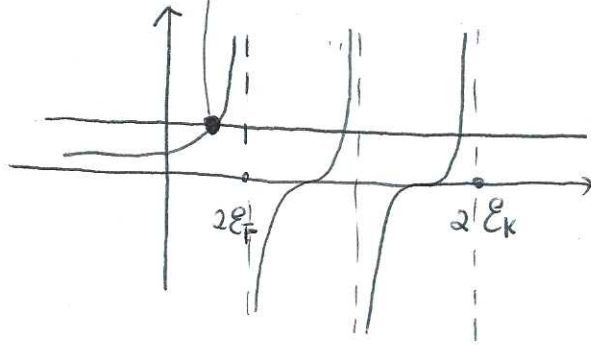
$$\Rightarrow \sum_{\mathbf{k}} \left[ (2E_{\mathbf{k}} + E_0) \underbrace{\alpha(\mathbf{k})}_U + u \sum_{\mathbf{k}'} \alpha(\mathbf{k}') \right] C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle = E \sum_{\mathbf{k}} \alpha(\mathbf{k}) C_{\mathbf{k}\uparrow}^{\dagger} C_{-\mathbf{k}\downarrow}^{\dagger} |F\rangle$$

i.e.  $(2E_{\mathbf{k}} + E_0) \alpha(\mathbf{k}) + u \sum_{\mathbf{k}'} \alpha(\mathbf{k}') = E \alpha(\mathbf{k})$

$$\Rightarrow \alpha(\mathbf{k}) \cdot = \frac{u}{E_0 + 2E_{\mathbf{k}} - E} \sum_{\mathbf{k}'} \alpha(\mathbf{k}')$$

$$\Rightarrow \frac{1}{u} = \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}} - (E - E_0)} \Rightarrow \frac{1}{u} = \sum_{\mathbf{k}} \frac{1}{-4E + 2E_{\mathbf{k}}}$$

bound state solution



Cooper pair binding  
energy!

$$\frac{1}{u} = N(0) \int_0^{\hbar\omega_D} dE \frac{1}{2E - \Delta E}$$

$$= \frac{N(0)}{2} \ln \frac{2\hbar\omega_D - \Delta E}{-\Delta E}$$

$$\Rightarrow \frac{2}{N(0)u} \approx \ln \frac{2\hbar\omega_D}{|\Delta E|}$$

$$\Delta E = 2\hbar\omega_D e^{-\frac{2}{N(0)u}}$$