HW 2

211A

Prob 2.1

a) $N = \frac{V}{(2\pi)^2}\int_0^{k_F} d\vec{k}^2 \Rightarrow n = \frac{\pi k_F^2}{(2\pi)^2} = \frac{k_F^2}{4\pi}$

For spin-1/2 electrons $\Rightarrow n = \frac{k_F^2}{2\pi}$

b) $\frac{1}{n} = \pi r_s^2 \Rightarrow r_s = \frac{\sqrt{2}}{k_f}$ for spin-1/2.

c) $g(\epsilon)\, d\epsilon = \frac{2}{(2\pi)^2}\cdot 2\pi\, k\, dk$, $\epsilon = \frac{\hbar^2 k^2}{2m}$

$g(\epsilon) = \frac{1}{2\pi}\frac{dk^2}{d\epsilon} = \frac{1}{2\pi}\frac{2\pi m}{\hbar^2} = \frac{m}{\hbar^2}$ for $\epsilon > 0$

$g(\epsilon) = 0$ for $\epsilon < 0$

d) Sommerfeld expansion: for any smooth function $H(\epsilon)$

$$\int_{-\infty}^{+\infty} H(\epsilon) f(\epsilon)\, d\epsilon = \int_{-\infty}^{\mu} H(\epsilon)\, d\epsilon + \sum_{n=1}^{\infty} (k_B T)^{2n} a_n \frac{d^{2n-1}}{d^{2n-1}} H(\epsilon)\Big|_{\epsilon=\mu}$$

where $f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T}+1}$ and $a_n = \frac{1}{(2n)!}\int_{-\infty}^{+\infty} dx \frac{e^x}{(e^x+1)^2}$.

for the density equation $n = \int_{-\infty}^{+\infty} g(\epsilon)\, d\epsilon \Rightarrow H(\epsilon) = g(\epsilon)$

$\Rightarrow$ all high other terms vanish $\Rightarrow$ small temperatures.

$n = \int_{-\infty}^{\mu(T)} g(\epsilon)\, d\epsilon \Rightarrow \mu(T) = \epsilon_F$ at

②

e) from 2.67 $n = \frac{m}{\pi\hbar^2}\int_0^{+\infty} \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1}$ define $x = \beta(\epsilon-\mu)$

$$\Rightarrow n = \frac{m}{\pi\hbar^2\beta}\int_{-\beta\mu}^{+\infty}\frac{dx}{e^x+1} \quad , \quad \int\frac{dx}{e^x+1} = \int\left(1-\frac{e^x}{e^x+1}\right)dx$$

$$= x - \int\frac{de^x}{e^x+1} = x - \ln(1+e^x)$$

$$= x - \ln e^x(1+e^{-x})$$

$$= -\ln(1+e^{-x})$$

$$= \frac{m}{\pi\hbar^2\beta}\left[\ln(1+e^{-x})\right]\Big|_{+\infty}^{-\beta\mu}$$

$$= \frac{m}{\pi\hbar^2\beta}\ln(1+e^{\beta\mu}) = \frac{m}{\pi\hbar^2\beta}\left[\beta\mu + \ln(1+e^{-\beta\mu})\right]$$

$$= \frac{m}{\pi\hbar^2}\left[\mu + \frac{1}{\beta}\ln(1+e^{-\beta\mu})\right]$$

at $T=0 \Rightarrow \quad n = \frac{m}{\pi\hbar^2}\mathcal{E}_F$

$T>0 \quad n = \frac{m}{\pi\hbar^2}\left[\mu + \frac{1}{\beta}\ln(1+e^{-\beta\mu})\right]$

$$\Rightarrow \boxed{\mathcal{E}_F = \mu + \frac{1}{\beta}\ln(1+e^{-\beta\mu})}$$

$$\Rightarrow \quad \mu - \mathcal{E}_F \approx \frac{1}{\beta}\ln(1+e^{-\beta\mathcal{E}_F}) \approx \frac{1}{\beta}\left[e^{-\beta\epsilon_F} - \frac{e^{-2\beta\epsilon_F}}{2} + \cdots\right]$$

f) The failure of Sommerfeld expansion is because

$\ln(1+e^{-\beta\mu})$ is not analytic function as $\beta \to +\infty$, which can be expanded as power series of $\frac{1}{\beta}$.

Prob 2

a) $$u = \frac{1}{V}\sum_k \frac{\epsilon}{e^{(\epsilon-\mu)/k_BT}+1} = \frac{1}{V}\sum_k f_k(T)\,\epsilon$$

$$\cancel{\frac{\partial u}{\partial T} = \frac{1}{V}\sum_k \frac{\epsilon\,(e^{(\epsilon-\mu)/k_BT})\left[\frac{d\mu}{dT}\frac{1}{k_BT} + \frac{\epsilon-\mu(T)}{k_BT^2}\right]}{(e^{(\epsilon-\mu)/k_BT}+1)^2}}$$

$$\left(\frac{\partial S}{\partial T}\right)_n = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_n = \frac{1}{V}\sum_k \frac{\epsilon_k}{T}\frac{d}{dT}f_k(T)$$

$$S(T) = \underset{\overset{\parallel}{0}}{S(0)} + \int_0^T dT' \frac{\partial S(T')}{\partial T'} = \frac{1}{V}\int_0^T dT' \sum_k \frac{\varepsilon_k}{T'}\frac{d}{dT'}f_k(T')$$

check $\frac{d}{dT}(f\ln f + (1-f)\ln(1-f))$

$$= \frac{d}{dT}f\cdot\ln f + \frac{f}{f}\frac{df}{dT} + (-)\frac{df}{dT}\ln(1-f) + \frac{(1-f)}{1-f}(-)\frac{df}{dT}$$

$$= \frac{d}{dT}f\ \ln\frac{f}{1-f} = \frac{d}{dT}f\ \ln e^{-\beta(\varepsilon_k-\mu)} = -\beta(\varepsilon_k-\mu)\frac{d}{dT}f$$

From $n = \frac{1}{V}\sum_k f(T) \Rightarrow \frac{dn}{dT} = \frac{1}{V}\sum_k \frac{d}{dT}f_k(T) = 0$

μ is a const for all k

$$\Rightarrow S(T) = \frac{1}{V}\sum_k \int_0^T dT' \left(\frac{\epsilon_k-\mu}{T'}\right)\frac{d}{dT'}f_k(T')$$

$$= -k_B\sum_k \int_0^T dT' \left(\frac{d}{dT'}(f_k\ln f_k + (1-f_k)\ln(1-f_k)\right)$$

$$= -k_B\sum_k \left[f_k\ln f_k + (1-f_k)\ln(1-f_k)\right]\Big|_0^T = -k_B\sum_k f_k\ln f_k + (1-f_k)\ln(1-f_k)$$

b) $$P = -[u - TS - n\mu]$$

$$u = \frac{1}{V}\sum_k \epsilon_k f_k(\epsilon)$$

$$TS = -\frac{1}{\beta V}\sum_k f_k\ln f_k + (1-f_k)\ln(1-f_k)$$

$$n = \frac{1}{V}\sum_k f_k$$

$$-(U - TS - \mu n) = -\frac{1}{\beta V} \sum_k \left[\beta(\varepsilon_k - \mu) f_k + f_k \ln f_k + (1-f_k)\ln(1-f_k)\right]$$

$$f_k = \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1} \qquad 1-f_k = \frac{e^{\beta(\varepsilon_k-\mu)}}{e^{\beta(\varepsilon_k-\mu)}+1} \qquad \text{defin } x = \beta(\varepsilon_k - \mu)$$

$\Rightarrow$ the term inside $\sum_k \to \dfrac{x}{e^x+1} + \dfrac{1}{e^x+1}\ln\dfrac{1}{e^x+1} + \dfrac{e^x}{e^x+1}\ln\dfrac{e^x}{e^x+1}$

$$= \frac{x}{e^x+1} + \ln\frac{1}{e^x+1} + \frac{e^x}{e^x+1}\ln e^x = \frac{x}{e^x+1}(1+e^x) + \ln\frac{1}{e^x+1}$$

$$= x - \ln(1+e^x) = \ln e^x/(1+e^x) = -\ln(1+e^{-x})$$

$$\Rightarrow p = \frac{1}{\beta V}\sum_k \ln(1+e^{-x}) = k_B T \int \frac{d^3k}{4\pi^3} \ln\left(1 + e^{-\frac{\frac{\hbar^2 k^2}{2m} - \mu}{k_B T}}\right).$$

$$p(\lambda\mu, \lambda T) = k_B \lambda T \int \frac{d^3k}{4\pi} \ln\left(1 + e^{-\frac{\frac{\hbar^2 k^2}{2m} - \lambda\mu}{\lambda k_B T}}\right)$$

$$= k_B \lambda T \underset{\lambda^{3/2}}{} \underbrace{\int \frac{d^3 \frac{k}{\sqrt{\lambda}}}{4\pi} \ln\left(1 + e^{-\left(\frac{\hbar^2 (\frac{k}{\sqrt{\lambda}})^2}{2m} - \mu\right)/k_B T}\right)} = \lambda^{5/2} p(\mu, T).$$

c)

$$dE = -p\,dV + T\,dS + \mu\,dN$$
$$dF = -p\,dV - S\,dT + \mu\,dN$$
$$d\Omega = -p\,dV - S\,dT - N\,d\mu$$

$$\Rightarrow -\left(\frac{\partial P}{\partial \mu}\right)_T = -\left(\frac{\partial N}{\partial V}\right) \Rightarrow \left(\frac{\partial P}{\partial \mu}\right)_T = n$$

$$\left(\frac{\partial P}{\partial T}\right)_\mu = \frac{\partial S}{\partial V} = s$$

d) $$P(\lambda\mu, \lambda T) = \lambda^{5/2} P(\mu, T)$$

$$\lambda \frac{dP}{d\lambda} = \frac{\partial P}{\partial(\lambda\mu)} \cdot \mu + \frac{\partial P}{\partial(\lambda T)} \cdot T = 5/2\, P(\lambda\mu, \lambda T)$$

set $\lambda=1$ $\Rightarrow$ $$\mu \cdot \left(\frac{\partial P}{\partial \mu}\right)_T + T \left(\frac{\partial P}{\partial T}\right)_\mu = \frac{5}{2} P(\mu, T)$$

$$\Rightarrow \boxed{n\mu + ST = \frac{5}{2} P} \Rightarrow P = -(U - TS - \mu n)$$

$$= -(U - \frac{5}{2} P)$$

$$\Rightarrow P = \frac{2}{3} U$$