HW1

$E_y = \pm i E_x$

Prob 4: $\vec{H} = H\hat{z}$, $\vec{E} = E_x e^{-i\omega t}\hat{x} \pm i E_x e^{-i\omega t}\hat{y}$

a) $$\dot{\vec{P}}(\omega) = -e\left(\vec{E}(\omega) + \frac{1}{mc}\vec{P}(\omega)\times\vec{H}\right) - \frac{1}{\tau}\vec{P}(\omega) = -i\omega\,\vec{P}(\omega)$$

where we assume that $\vec{P}(t) = \vec{P}(\omega)\,e^{-i\omega t}$ following the same frequency.

1. $$\left(\frac{1}{\tau} - i\omega\right)P_x(\omega) = -e\left(E_x(\omega) + \frac{1}{mc}P_y(\omega)H_z\right)$$

2. $$\left(\frac{1}{\tau} - i\omega\right)P_y(\omega) = -e\left(\pm i E_x(\omega) - \frac{1}{mc}P_x(\omega)H_z\right)$$

3. $$\left(\frac{1}{\tau} - i\omega\right)P_z(\omega) = 0$$

(3) $\Rightarrow$ $P_z(\omega) = 0$, or $j_z = 0$

$$(1 - i\omega\tau)P_x(\omega) + \frac{e\tau}{mc}H\,P_y(\omega) = -e E_x(\omega)$$

$$-\frac{e\tau H}{mc}P_x(\omega) + (1 - i\omega\tau)P_y(\omega) = \mp i e E_x(\omega)$$

$$\Rightarrow P_x = \frac{-e\tau}{1 - i(\omega \mp \omega_c)\tau}E_x \qquad \text{where } \omega_c = \frac{eH}{mc}$$

$$P_y = \frac{\mp i e\tau}{1 - i(\omega \mp \omega_c)\tau}E_x$$

$$\left.\begin{aligned} j_x &= -en\frac{P_x}{m} \\ j_y &= -en\frac{P_y}{m} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} J_x &= \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau}E_x \\ J_y &= \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau}E_y = \pm i J_x \end{aligned}$$

b) Maxwell equations $\Rightarrow$

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E} + \nabla(\nabla\cdot\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{B}$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi\sigma}{c}\vec{E} + \frac{1}{c}\frac{\partial E}{\partial t}$$

Consider the case of no charge density, i.e. $\nabla\cdot\vec{E}=0$, and consider the frequency of ω, $\Rightarrow$ $k^2\vec{E}_{(\omega)} = +i\omega\frac{4\pi}{c^2}\sigma\vec{E}_{(\omega)} + \frac{\omega^2}{c^2}\vec{E}(\omega)$

$$\Rightarrow k^2c^2 = \varepsilon\omega^2 + i\omega 4\pi\sigma = \omega^2\left[1 + \frac{4\pi i}{\omega}\sigma\right]$$

where $\sigma = \dfrac{\sigma_0}{1 - i(\omega\mp\omega_c)\tau}$

$$\Rightarrow k^2c^2 = \omega^2\left[1 + \frac{4\pi i}{\omega}\frac{ne^2\tau}{m}\frac{1}{1-i(\omega\mp\omega_c)\tau}\right]$$

$$= \omega^2\Big[\underbrace{1 - \frac{\omega_p^2}{\omega}\frac{1}{\omega\mp\omega_c + i/\tau}}_{\varepsilon(\omega)}\Big], \quad \text{where } \omega_p^2 = \frac{4\pi ne^2}{m}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{1}{\omega\mp\omega_c + i/\tau}$$

c) $$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{\omega-\omega_c - i/\tau}{(\omega-\omega_c)^2 + 1/\tau^2} \quad \text{for } E_y = iE_x.$$

$$\mathrm{Re}\,\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{(\omega-\omega_c)\tau^2}{(\omega-\omega_c)^2\tau^2 + 1}$$

$$\mathrm{Im}\,\varepsilon(\omega) = \frac{\omega_p^2}{\omega}\frac{\tau}{(\omega-\omega_c)^2\tau^2+1}$$

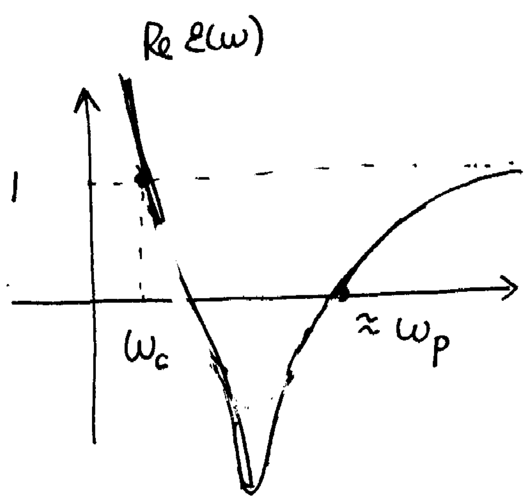
Considering the fact that $\omega_p/\omega_c \gg 1$, and $\omega_c \tau \gg 1$,

at $\omega > \omega_p$, we approximate $\varepsilon(\omega) = 1 - \frac{\omega_p}{\omega} \frac{1}{\frac{\omega}{\omega_p} - \frac{\omega_c}{\omega_p} + \frac{i}{\omega_p \tau}} \approx 1 - \frac{\omega_p^2}{\omega^2}$

$\Rightarrow 1 > \varepsilon(\omega) > 0$ i.e $k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}$ is real. and $\frac{\omega}{k} > c$

at $\omega < \omega_c$ $\quad \varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega} \frac{1}{\omega_c - \omega - i/\tau} = 1 + \frac{\omega_p^2}{\omega\omega_c} \frac{1}{1 - \frac{\omega}{\omega_c} - \frac{i}{\omega_c \tau}}$

$\approx 1 + \frac{\omega_p^2}{\omega\omega_c} \frac{1}{1 - \frac{\omega}{\omega_c}} > 1 \quad \Rightarrow k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)} \Rightarrow \frac{\omega}{k} < c.$



d) at $\omega \ll \omega_c$

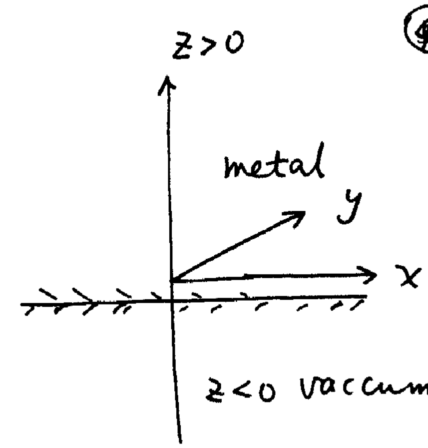
$$\varepsilon(\omega) \approx 1 + \frac{\omega_p^2}{\omega\omega_c} \approx \frac{\omega_p^2}{\omega\omega_c}$$

$$\Rightarrow \omega = \frac{kc}{\sqrt{\varepsilon(\omega)}} = \frac{kc}{\omega_p}\sqrt{\omega\omega_c}$$

$$\Rightarrow \omega = \omega_c \left(\frac{kc}{\omega_p}\right)^2$$

for $\lambda = 1\,\text{cm}$, $T = 10\,\text{kG}$, $c = 3\times10^{10}\,\text{cm/s}$ $\quad e = 1.6\times10^{-19}\,C$

$$f = \frac{\omega}{2\pi} = \frac{eH}{mc}\left(\frac{kc}{\frac{4\pi n e^2}{m}}\right)^2 \frac{1}{2\pi} = \frac{Hc}{8\pi^2 ne} k^2 = \frac{Hc}{2ne}\frac{1}{\lambda^2} \approx 3.1\,\text{Hz}$$



Prob 1.5:

try the solution of surface plasmons

$z>0 \quad E_x = A e^{iqx-Kz}, \quad E_y = 0, \quad E_z = B e^{iqx} e^{-Kz}$

$z<0 \quad E_x = C e^{iqx} e^{K'z}, \quad E_y = 0, \quad E_z = D e^{iqx} e^{K'z}$

inside the metal $\quad \varepsilon(\omega) = 1 + \frac{i4\pi\sigma(\omega)}{\omega} = 1 + \frac{i4\pi}{\omega}\frac{\sigma_0}{1-i\omega\tau}$

the boundary conditions: $E_\parallel$ continuous $\Rightarrow A = C$

$\varepsilon E_\perp$ continuous $\Rightarrow \left(1 + i\frac{4\pi}{\omega}\frac{\sigma_0}{1-i\omega\tau}\right) B = D$

the condition $\nabla\cdot E = 0 \Rightarrow$ for $z>0$ $\boxed{iAq = KB}$

for $z<0$ $\boxed{iqC = -K'D}$

$$\text{for } z>0 \quad -\nabla^2 E = \epsilon\frac{\omega^2}{c^2}E \Rightarrow q^2 - K^2 = \frac{\omega^2}{c^2}\epsilon$$

$$z<0 \quad -\nabla^2 E = \frac{\epsilon\omega^2}{c}E \Rightarrow q^2 - K'^2 = \frac{\omega^2}{c^2}$$

Solve these equations

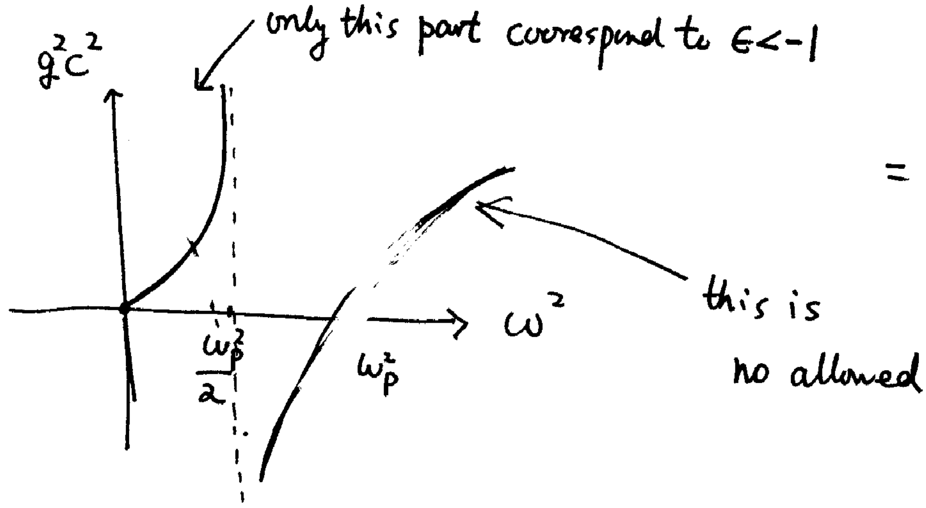
$$\left.\begin{aligned} K^2 - K'^2 &= \frac{\omega^2}{c^2}(1-\epsilon(\omega)) \\ \frac{K}{K'} &= \frac{iqA}{B}\frac{(-D)}{iqC} = -\frac{AD}{BC} = -\epsilon(\omega) \end{aligned}\right\} \Rightarrow \begin{aligned} K^2 &= -\left(\frac{\omega}{c}\right)^2 \frac{\epsilon(\omega)^2}{\epsilon(\omega)+1} \\ K'^2 &= -\left(\frac{\omega}{c}\right)^2 \frac{1}{\epsilon(\omega)+1} \end{aligned}$$

We need the region that $\epsilon(\omega) < -1$, (we neglect the imaginary part of ε)

$$\Rightarrow \quad K = \frac{\omega}{c}\frac{-\epsilon(\omega)}{\sqrt{-(1+\epsilon(\omega))}}, \qquad K' = \frac{\omega}{c}\frac{1}{\sqrt{-(1+\epsilon(\omega))}}$$

$$q^2 = K^2 + \frac{\omega^2}{c^2}\epsilon = \frac{\omega^2}{c^2}\frac{\epsilon}{1+\epsilon}$$

b) when $\omega\tau \gg 1 \Rightarrow \epsilon = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow q^2c^2 = \omega^2 \frac{1-\frac{\omega_p^2}{\omega^2}}{2-\frac{\omega_p^2}{\omega^2}}$



$$= \frac{\omega^2}{2}\frac{\omega^2-\omega_p^2}{\omega^2-\frac{\omega_p^2}{2}}$$

(for $\omega^2 < \frac{\omega_p^2}{2}$).

c) in the limit $q^2c^2 \gg \omega^2 \Rightarrow \frac{q^2c^2}{\omega^2/2} = \frac{\omega^2-\omega_p^2}{\omega^2-\omega_p^2/2} \gg 1$

$\Rightarrow \omega = \omega_p/\sqrt{2}$. Assume that $\omega^2 = \frac{\omega_p^2}{2} - \delta^2$

$$\Rightarrow K' = \frac{\omega}{c}\frac{\sqrt{\omega_p^2/2}}{\sqrt{2}\left(\sqrt{-\omega^2+\frac{\omega_p^2}{2}}\right)} = \frac{\omega_p^2}{2c\,\delta} \approx \frac{\omega_p}{\delta}\left(\frac{\omega_p}{2c}\right) \gg K_p$$

$\therefore$ K' is much largen that the wave length of light at plasamon frequency! $\frac{K}{K'} = -\epsilon(\omega) \sim 1$. the decay in the vaccum side is also equally fast.

from $$K' = \frac{\omega_p^2}{\sqrt{2}\, c\, \delta}$$ and $$\frac{q^2 c^2}{\omega_p^2/2} = \frac{\frac{\omega_p^2}{2}}{\delta^2} \Rightarrow qc = \frac{\frac{\omega_p^2}{2}}{\delta}$$

$$\Rightarrow K \simeq K' = q \Rightarrow \begin{array}{l} D = -iC \\ B = iA \end{array} \Rightarrow$$ circulary polarized !

polarization in the xz plane with opposite polarizibity at $z>0$, $z<0$.