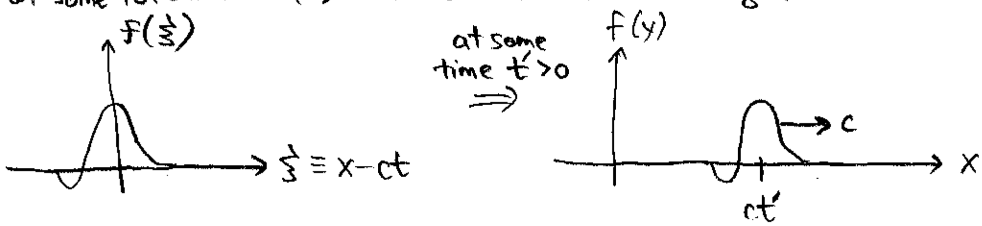
PHYSICS 110B

HW #6

16.12

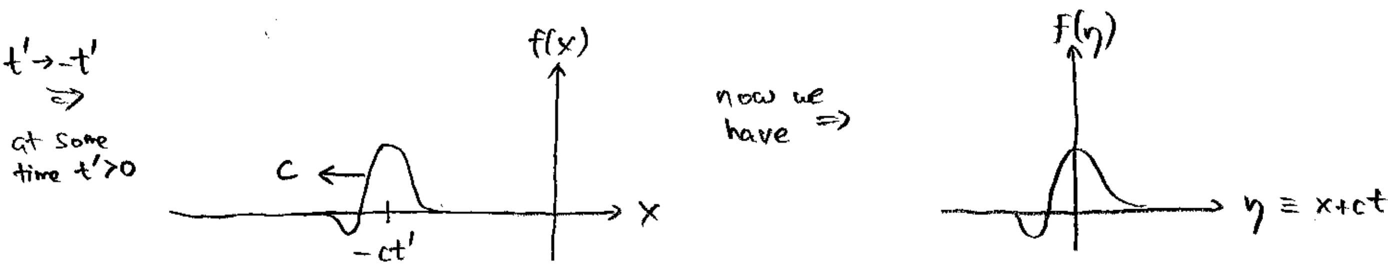
We start w/ some function $f(\xi)$ localized around the origin:



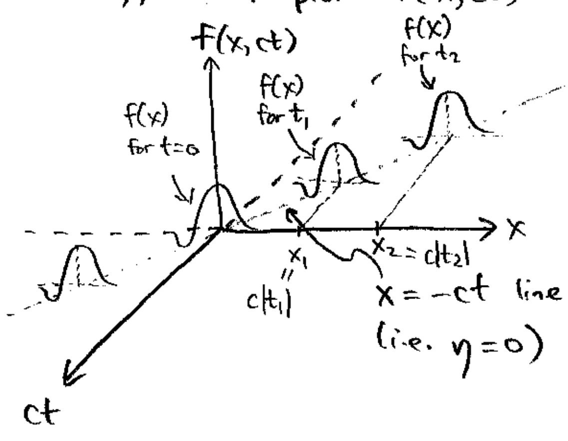
Ostensibly, $f(x-ct)$ is a right-moving wave of speed c.

a) We obtain the wave given by the function $f(\underbrace{x+ct}_{\equiv \eta})$ by performing

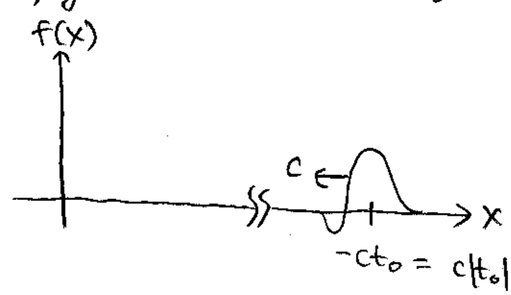
the time inversion: $t \to -t$



Holistically, we can plot $f(x, ct)$:



If we want wave given by $f(x+ct)$ for large negative time t_0, just take $f(x)$ for t_0:



Thus, $f(x+ct_0)$ is a wave far to the right (at $x = c|t_0|$) traveling at speed c to the left.

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b) We wish to solve for the motion $u(x,t)$ of the wave previously discussed now along a semi-infinite string satisfying the boundary condition:

$u(0,t)=0$ for all t; Obviously, our original $f(x+ct)$ does not satisfy this B.C.

We use the method of images to construct the unique solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{or,} \qquad \frac{\partial}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}=0$$

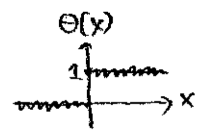
in terms of coordinates

$$\begin{cases} \eta = x+ct \\ \xi = x-ct \end{cases}$$

We are given the function $u(x,t) = f(\underbrace{x+ct}_{\eta}) - f(\underbrace{-x+ct}_{-\xi})$.

Since the wave must reside on the semi-infinite string, let us include the Heaviside step function as a factor:

$$\Rightarrow u(x,t) = \Theta(x)\left[f(x+ct) - f(-x+ct)\right], \text{ where } \Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x<0 \end{cases}$$

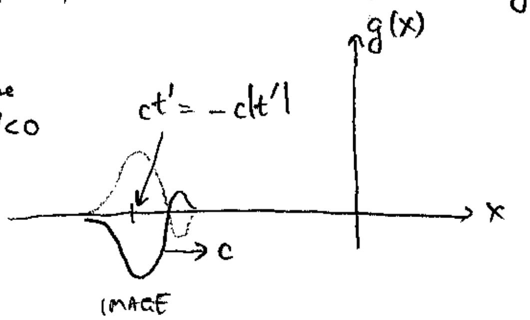


The second term inside is obtained by two successive inversions of the first term:

$$\begin{cases} x \to -x, & \text{reflection of } f(x) \text{ about } x=0 \\ f \to -f, & \text{" " " " } f=0 \end{cases}$$

Thus, we have for $-f(-x+ct) \equiv g(x,t)$

at some time $t'<0$



Thus, $-f(-x+ct)$ is a right-moving wave of speed c. It is ostensibly the twice-reflected "image" of the wave $f(x+ct)$ of part (a). Since we are given that $f(\xi) = f(x-ct)$ is a solution to the wave equation (ignoring BC for now), so is $f(-\eta) = f(-x+ct)$. Thus, $g(x,t) = -f(-x+ct)$ is also a solution.

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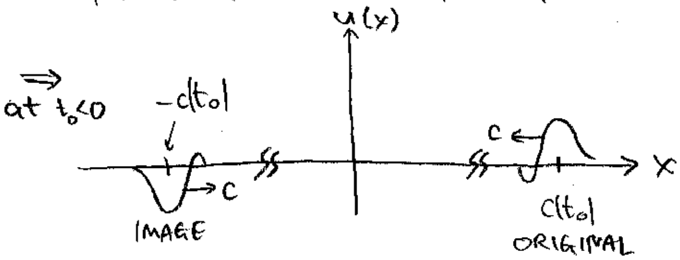
b) By linearity of the wave equation, we have the superposition :

$\Rightarrow \; u(x,t) = \Theta(x)\,[f(x+ct) - f(-x+ct)]$

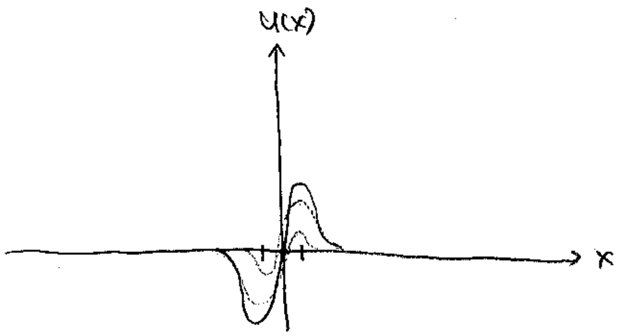
which uniquely solves the wave equation for the BC : $u(x=0,t)=0$, all t

$\Rightarrow \; \boxed{u(0,t) = f(ct) - f(ct) = 0}$. We only need to specify some initial condition.

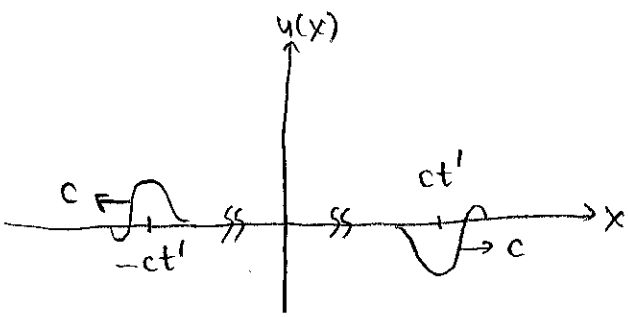
c) The motion on the semi-infinite string can be described as follows for the initial condition from part (a) : $u(x,t_0) = f(x+ct_0)$



Ignoring the image term now suppressed by the step function for $t_0<0$, the wave travels leftward from far right of the origin.



For t near zero, the original wave starts to interfere w/ the image wave. Notice that the boundary condition $u(0,t)=0$ is always satisfied.



Eventually for $t' \gg 0$, the image wave is the solution and the original wave is suppressed. (Upon reflection at the origin, the wave is inverted and travels in the opposite direction at speed c)

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We are given the stress tensor $\overleftrightarrow{\Sigma}$ (a real symmetric matrix) at a pt. P of the continuous medium. By solving the eigenvalue problem $\det(\overleftrightarrow{\Sigma} - \lambda \overleftrightarrow{\mathbb{1}}) = 0$, we can find the principal stress axes at P (as we did for the moment of inertia tensor, $\overleftrightarrow{I}$) and using this basis, write the stress tensor in diagonal form:

$$\Rightarrow \overleftrightarrow{\Sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

, where we have rotated the original Cartesian coordinate axes to coincide with the principal stress axes whose directions are now $\hat{x}, \hat{y}, \hat{z}$

If we consider a surface force on a small element of area dA normal to the x-axis of this basis:

$$\Rightarrow d\underline{A} = \begin{pmatrix} dA \\ 0 \\ 0 \end{pmatrix} \; ; \quad \underline{F} = \overleftrightarrow{\Sigma}\, d\underline{A} = dA \begin{pmatrix} \sigma_{11} \\ 0 \\ 0 \end{pmatrix} = \sigma_{11}\, dA\, \hat{x}$$

We have found that the surface force on the surface normal to $\hat{x}$ is also normal to the surface (i.e. $\underline{F} \parallel d\underline{A}$, or $\underline{F} \parallel \hat{x}$). We can repeat the exact same argument for the $\hat{y}$ and $\hat{z}$-directions.

$\therefore$ The surface force on any surface normal to one of the principal stress axis directions is exactly normal to the surface.

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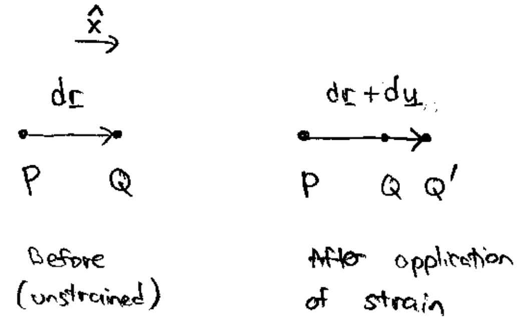

We are given the strain tensor $\overleftrightarrow{E}$ at a pt. P in the continuous solid. Initially, the pt. Q is a distance dr away from pt. P. Since we assume that applied strain leaves P fixed and neighborhood of P unrotated, the displacement $d\underline{u}$ of Q is in the same direction as $d\underline{r}$ (chosen to be $\hat{x}$-direction).

$$d\underline{r} = \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix}$$

a) Generally, we obtain the displacement from:

$$d\underline{u} = \overleftrightarrow{D}\, d\underline{r} \qquad (16.75)$$

where $\overleftrightarrow{D} \equiv$ derivatives tensor, which can be decomposed into an asymmetric ($\overleftrightarrow{A}$) and symmetric ($\overleftrightarrow{E}$) part:

$$\overleftrightarrow{D} = \overleftrightarrow{A} + \overleftrightarrow{E} \qquad (16.80)$$

Since the asymmetric part $\overleftrightarrow{A}$ represents rigid rotations, it vanishes for this problem. not same as shearing!

Hence, $d\underline{u} = \overleftrightarrow{E}\, d\underline{r} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon_{11}\, dx \\ \epsilon_{21}\, dx \\ \epsilon_{31}\, dx \end{pmatrix}$

But since we are given that pts. in the neighborhood of P are left unrotated, $d\underline{u} \parallel \hat{x}$ must hold, thus $\epsilon_{21} = \epsilon_{31} = 0$ (no shearing strains) and we have:

$$d\underline{u} = \epsilon_{11}\, dx\, \hat{x}$$

Thus, $d\underline{r} + d\underline{u} = (1 + \epsilon_{11})\, dx\, \hat{x}$ and so the x-axis near P is stretched by a factor of $\boxed{1+\epsilon_{11}}$

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b) We perform the exact same analysis in part (a) for the $\hat{y}$ and $\hat{z}$-directions to obtain:

$$\begin{cases} d\underline{x} + d\underline{u}_x = (1+\epsilon_{11})\,dx\ \hat{x} \\ d\underline{y} + d\underline{u}_y = (1+\epsilon_{22})\,dy\ \hat{y} \\ d\underline{z} + d\underline{u}_z = (1+\epsilon_{33})\,dz\ \hat{z} \end{cases}$$

where we have relabeled $d\underline{r}$ and $d\underline{u}$ for clarity

Initially, (before application of strain) the small volume around P is given by:

$$V = dx\,dy\,dz$$

The new volume after application of strain is given by:

$$V' = (1+\epsilon_{11})\,dx\,(1+\epsilon_{22})\,dy\,(1+\epsilon_{33})\,dz$$

$$\simeq (1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33})V$$, where we have ignored terms of higher order $\mathcal{O}(\epsilon^2)$

Thus, $$\frac{dV}{V} = \frac{V'-V}{V} = \boxed{\epsilon_{11}+\epsilon_{22}+\epsilon_{33} \equiv \operatorname{tr}\overleftrightarrow{E}}$$

We can further decompose $\overleftrightarrow{E}$ into:

$$\overleftrightarrow{E} = e\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{E}'\,,$$ where $e \equiv \frac{1}{3}\operatorname{tr}\overleftrightarrow{E} = \frac{1}{3}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33})$, the average stretch of the strain tensor, $\overleftrightarrow{E}$

so that the first term represents a pure dilatation (i.e. spherical stretching in which each of the stretching directions are stretched by the same amount e) and the second term represents corrections about the pure dilatation.

Generally, $\overleftrightarrow{E}'$ contains both stretching (diagonal elements) and shearing (off-diagonal) elements but for this problem, off-diagonal elements vanish due to the non-rotation condition.

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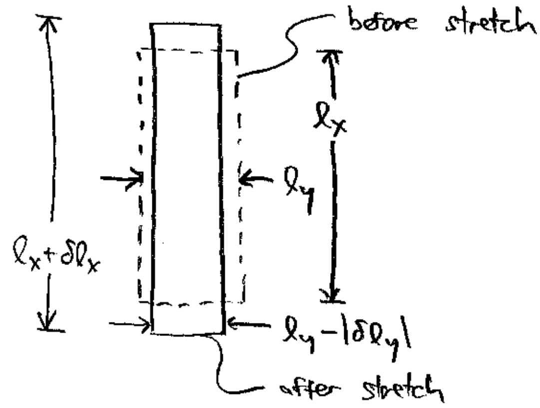
By definition, Poisson's ratio, $\nu = \dfrac{\text{transverse fractional contraction}}{\text{longitudinal fractional stretch}} = \dfrac{-\delta l_y / l_y}{\delta l_x / l_x} \geq 0$

a) for the wire pulled taut in the x-direction:

the ratios we seek are picked out of the strain tensor $\overleftrightarrow{E}$:



$$\Rightarrow \overleftrightarrow{E} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

where $\epsilon_{11} = \dfrac{\delta l_x}{\delta l_x}$ and $\epsilon_{22} = \dfrac{\delta l_y}{\delta l_y}$

Hence, $\boxed{\nu = -\dfrac{\epsilon_{22}}{\epsilon_{11}}}$

b) Let us find all elements of the strain tensor, $\overleftrightarrow{E}$.

From (16.95):

$$\overleftrightarrow{E} = \frac{1}{3\alpha\beta}\left[3\alpha \overleftrightarrow{\Sigma} - (\alpha-\beta)(\mathrm{tr}\,\overleftrightarrow{\Sigma})\,\overleftrightarrow{\mathbb{1}}\right]$$

where as given in problem 16.27 we have

$$\overleftrightarrow{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$, due to application of a pure tension along the x-axis. Hence, $\mathrm{tr}\,\overleftrightarrow{\Sigma} = \sigma_{11}$

$$\Rightarrow \overleftrightarrow{E} = \frac{\sigma_{11}}{3\alpha\beta}\begin{bmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

$$\Rightarrow \nu = \frac{\alpha-\beta}{2\alpha+\beta} = \boxed{\frac{3BM - 2SM}{6BM + 2SM}}$$, where we have used the definitions of the bulk and shear modula:

$\alpha = 3BM$ (16.97)

$\beta = 2SM$ (16.99)

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c) Poisson's ratio for 5 materials:

$\Rightarrow$

Material	BM	SM	ν
Iron	90	40	.31
Steel	140	80	.26
Sandstone	17	6	.34
Perovskite	270	150	.27
Water	2.2	0	.50

As we can see, for the limit $BM \gg SM$,

$\Rightarrow \nu = \dfrac{3BM - \cancel{2SM}}{6BM + \cancel{2SM}} \simeq 1/2$, which matches what we obtained for water.

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