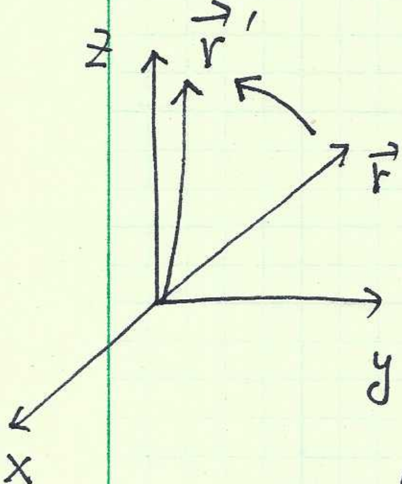
A review of linear algebra

§1. Orthogonal matrix — ratation



$$\vec{r}' = T\,\vec{r}$$

or in terms of components

$$r'_\alpha = T_{\alpha\beta}\, r_\beta$$

① r' and r are vectors, and T is 3X3 matrix

② repeated indices mean summation.

or
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

rotatation maintains the norm of r, or $\vec{r}'\cdot\vec{r}' = \vec{r}\cdot\vec{r}$

$$\Rightarrow r'_\alpha r'_\alpha = T_{\alpha\beta} r_\beta T_{\alpha\beta'} r_{\beta'} = r_\beta (T^t_{\beta\alpha} T_{\alpha\beta'}) r_{\beta'} = r_\beta\, \delta_{\beta\beta'}\, r_{\beta'}$$

$\delta_{\beta\beta'}$: Koronick δ-symbol

$$= \begin{cases} 1 & \text{if } \beta = \beta' \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow T^t_{\beta\alpha} T_{\alpha\beta'} = \delta_{\beta\beta'}$$

or $\boxed{T^t\, T = I}$ ← orthogonal matrix

t: means transpose

Orthogonal matrix is the basic tool to describle rotation.

Q: how many degree of freedom to describle a rotation?

T has $3 \times 3 = 9$ real variables

$T^t T$ is a symmetric matrix $\Rightarrow$ $T^t T = I$ means 6 independent constraints

$\Rightarrow$ $9 - 6 = 3$ degrees of freedom.

§ Example: rotation around z-axis at angle θ_{12} $\begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} & 0 \\ \sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

x-axis at angle θ_{23}: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & -\sin\theta_{23} \\ 0 & \sin\theta_{23} & \cos\theta_{23} \end{pmatrix}$

y-axis at angle θ_{31}: $\begin{pmatrix} \cos\theta_{31} & 0 & \sin\theta_{31} \\ 0 & 1 & 0 \\ -\sin\theta_{31} & 0 & \cos\theta_{31} \end{pmatrix}$

§2. Generators of 3D rotation

Consider infinitesimal rotation $T_{\alpha\beta} \simeq \delta_{\alpha\beta} + M_{\alpha\beta} \leftarrow$ small

check $T^t T = I \Rightarrow M^t = -M \leftarrow$ anti-symmetric

we parameterize $M = \begin{pmatrix} 0 & -\theta_{12} & \theta_{31} \\ \theta_{12} & 0 & -\theta_{23} \\ -\theta_{31} & \theta_{23} & 0 \end{pmatrix} = \theta_{12} L_{12} + \theta_{23} L_{23} + \theta_{31} L_{31}$

$\Rightarrow L_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad L_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad L_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Please check $(L_{12})_{ab} = -\epsilon_{3ab}$. $(L_{23})_{ab} = -\epsilon_{1ab}$, $(L_{31})_{ab} = -\epsilon_{2ab}$.

ϵ_{abc} is the rank-3 fully anti-symmetric tensor

$$\epsilon_{abc} = \begin{cases} 1 & \text{for } abc = 123,\ 231,\ 312 : \text{ even permutations} \\ -1 & \phantom{\text{for } abc} = 132,\ 213,\ 321 \quad \text{odd permutations.} \end{cases}$$

Exercise: define $L_z = iL_{12} = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $L_x = iL_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}$

$L_y = iL_{31} = \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$.

Prove:

$$\begin{cases} L_x L_y - L_y L_x = iL_z \\ L_y L_z - L_z L_y = iL_x \\ L_z L_x - L_x L_z = iL_y \end{cases}$$

or $L_a L_b - L_b L_a = i\epsilon_{abc} L_c$

or $\boxed{[L_a, L_b] = i\epsilon_{abc} L_c}$

angular momentum commutation

you will learn it in QM.

§3 Definition of angular velocity ω.

Suppose from $t \to t + \Delta t$, the rotation of a body is described by an infinitesmall rotation

$$T_{\alpha\beta}(t;\Delta t) = \delta_{\alpha\beta} + \theta_{12} L_{12} + \theta_{23} L_{23} + \theta_{31} L_{31}$$

we define angular velocity $\omega_{12} = \frac{\Delta\theta_{12}}{\Delta t}$, $\omega_{23} = \frac{\Delta\theta_{23}}{\Delta t}$, $\omega_{31} = \frac{\Delta\theta_{31}}{\Delta t}$

and take the limit $\Delta t \to 0$.

Thus rigorously speaking, angular velocity is not an ordinary vector, but a rank-2 anti-symmetric tensor.

In 3-dimensions, for convenience, we often use

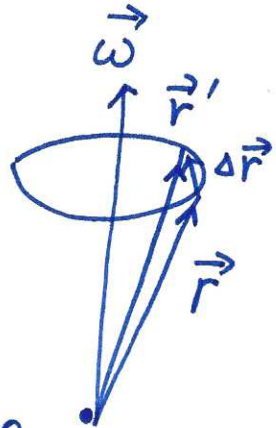
$$\omega_a = \frac{1}{2}\epsilon_{abc}\,\omega_{bc}$$ with definition $\omega_{21} = -\omega_{12}$, $\omega_{32} = -\omega_{23}$, $\omega_{13} = -\omega_{31}$

$$\Rightarrow \omega_1 = \omega_{23}, \quad \omega_2 = \omega_{31}, \quad \omega_3 = \omega_{12}$$

then we put $\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ as a 3-component of a 3-vector.

Lect 2. Rotating frame

§1. linear velocity v.s. angular velocity



$$r'_\alpha = T_{\alpha\beta}\, r_\beta = \left[\delta_{\alpha\beta} + \frac{\Delta\theta_{ab}}{2}\left(L_{ab}\right)_{\alpha\beta}\right] r_\beta$$

we symmetrize $\theta_{21} = -\theta_{12}$, $\theta_{32} = -\theta_{23}$, $\theta_{13} = -\theta_{31}$
by defining $L_{21} = -L_{12}$ $L_{32} = -L_{23}$ $L_{13} = -L_{31}$

$$\Rightarrow \quad r'_\alpha - r_\alpha = \frac{\Delta\theta_{ab}}{2}\left(L_{ab}\right)_{\alpha\beta} r_\beta$$

$$\Rightarrow \quad \frac{\Delta r_\alpha}{\Delta t} = \frac{\Delta\theta_{12}}{\Delta t}\left(L_{12}\right)_{\alpha\beta} r_\beta + \frac{\Delta\theta_{23}}{\Delta t}\left(L_{23}\right)_{\alpha\beta} r_\beta + \left(\frac{\Delta\theta_{31}}{\Delta t}\right)\left(L_{31}\right)_{\alpha\beta} r_\beta$$

$$= -\left[\omega_3\, \epsilon_{3\alpha\beta}\, r_\beta + \omega_1\, \epsilon_{1\alpha\beta}\, r_\beta + \omega_2\, \epsilon_{2\alpha\beta}\, r_\beta\right]$$

$$= \epsilon_{\alpha\beta\gamma}\, \omega_\beta\, r_\gamma \quad \Rightarrow \quad v_\alpha = \epsilon_{\alpha\beta\gamma}\, \omega_\beta\, r_\gamma \quad \text{i.e. } \boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

generally $$\boxed{\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}}$$

$$\boxed{\begin{aligned} \vec{r}_{CA} &= \vec{r}_{CB} + \vec{r}_{BA} \\ d\vec{r}_{CA}/dt &= \frac{d\vec{r}_{CB}}{dt} + \frac{d\vec{r}_{BA}}{dt} \\ \vec{\omega}_{CA} \times \vec{r} &= \left(\vec{\omega}_{CB} + \vec{\omega}_{BA}\right) \times \vec{r} \end{aligned}}$$

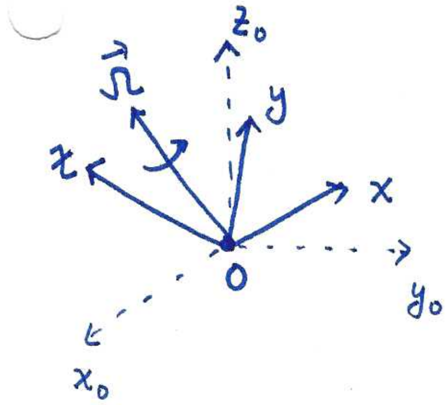
★ addition of angular velocities:

Suppose frame B is rotating with $\vec{\omega}_{BA}$ respect to frame A,

body C is rotating with $\vec{\omega}_{CB}$, respect to frame B,

then body C is rotating at $\boxed{\vec{\omega}_{CA} = \vec{\omega}_{CB} + \vec{\omega}_{BA}}$ respect to frame A.

§ 2. time-derivatives in a rotating frame



S_0: $x_0 y_0 z_0$ — inertial fixed frame

S: xyz — rotating frame with $\vec{\Omega}$ respect to

$e_1\, e_2, e_3$ unit ~~a~~ vectors along x, y, z axes in the rotating frame.

$$\text{vector } \vec{Q} = Q_1 \vec{e}_1 + Q_2 \vec{e}_2 + Q_3 \vec{e}_3$$

in frame S, $\vec{e}_{1,2,3}$ are const $\Rightarrow$ $\left(\dfrac{d\vec{Q}}{dt}\right)_S = \sum_i \dfrac{dQ_i}{dt} \vec{e}_i$

in the frame S_0, $\vec{e}_{123}$ are rotating $\dfrac{d\vec{e}_i}{dt} = \vec{\Omega} \times \vec{e}_i$

$$\Rightarrow \left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt} \vec{e}_i + \sum Q_i \underbrace{\frac{d\vec{e}_i}{dt}}_{\vec{\Omega} \times \vec{e}_i} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$$

§ 4: Newton's law in rotating frame:

$$\boxed{\begin{array}{l}\text{Newton's law in the inertial frame} \\ m\left(\dfrac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}\end{array}}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}$$

$$\left(\frac{d}{dt}\right)_{s_0}\left(\frac{d}{dt}\right)_{s_0}\vec{r} = \left(\frac{d}{dt}\right)_{s_0}\left(\frac{d\vec{r}}{dt}\right)_s + \left(\frac{d}{dt}\right)_{s_0}(\vec{\Omega}\times\vec{r})$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_s + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_s + \underbrace{\left(\frac{d}{dt}\right)_s(\vec{\Omega}\times\vec{r})}_{=\ \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_s} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_s + 2\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_s + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\Rightarrow\quad m\left(\frac{d^2\vec{r}}{dt^2}\right)_s = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{s_0} - 2m\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_s - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

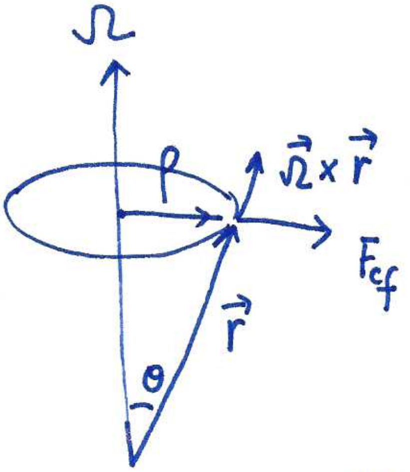
$$\boxed{m\ddot{\vec{r}}_s = \vec{F} + \underbrace{2m\dot{\vec{r}}_s\times\vec{\Omega}}_{\text{Coriolis force}} + \underbrace{m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}_{\text{centrifugal force}}}$$

§5. Centrifugal force

$$F_{cor} \sim m v\Omega \qquad F_{cf} \sim m r\Omega^2 \Rightarrow \frac{F_{cor}}{F_{cf}} \sim \frac{v}{r\Omega}$$

For earth $r\Omega \sim \dfrac{4\times10^4\,\text{km}}{24\,\text{h}} \sim 1.6\times10^3\,\text{km/h} \sim 1000\,\text{mil/h}$

where $v < 1.6\times10^3\,\text{km/h} \simeq 500\,\text{m/s}$, Corilis force is not important.



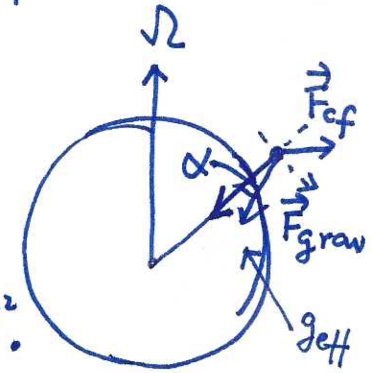
$$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

its direction is along the radial direction $\hat{\rho}$

$$|F_{cf}| = m\Omega r \sin\theta \, \Omega = m\Omega^2 \rho .$$

Free fall acceleration:

$$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} = m g_0 \underset{(-\hat{r})}{} + m\Omega^2 R \sin\theta \, \hat{\rho}$$



$$\vec{g}_{eff} = g_0(-\hat{r}) + \Omega^2 R \sin\theta \, \hat{\rho}$$

$$\Omega^2 R = 0.034 \, m/s^2 \sim 0.3\% \text{ of } g = 10 m/s^2 .$$

$$g_{tang} = \Omega^2 R \sin\theta \cos\theta \, \hat{e}_\theta$$

thus $\vec{g}_{eff}$ is not exactly normal to the surface, but slightly toward the equator $\quad \alpha = \frac{g_{tang}}{g} \simeq \frac{\Omega^2 R}{2g} \sin 2\theta$

$$\simeq 0.1^\circ \sin 2\theta .$$