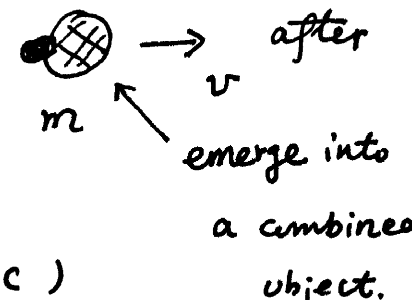
Lect 17. Relativity (III)

§ Collisions — application of momentum-energy conservation.

Example 1:

$$P^{\mu}_{a,i} = (m_a \gamma_a v_a, 0, 0, m_a \gamma_a c^2)$$

$$P^{\mu}_{b\,i} = (0, 0, 0, m_b c^2)$$



$$P_f = (m\gamma v, 0, 0, m\gamma c^2)$$

$$= P_{a,i} + P_{b_i} = (m_a \gamma_a v_a, 0, 0, m_a \gamma_a c + m_b c)$$

$$\Rightarrow \quad m^2 c^2 \gamma^2 [1 - (\tfrac{v}{c})^2] = (m_a \gamma_a c + m_b c)^2 - m_a \gamma_a v_a^2$$

$$(mc)^2 = (m_a c)^2 + (m_b c)^2 + 2 m_a m_b \gamma_a c^2$$

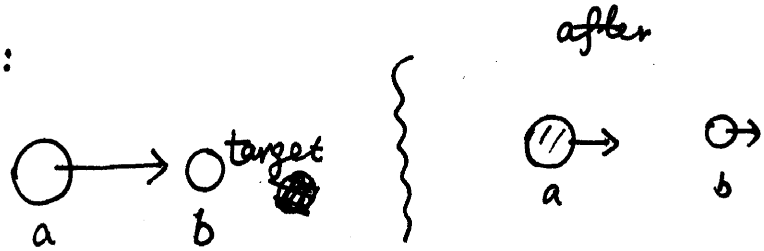
$$m = (m_a^2 + m_b^2 + 2 m_a m_b \gamma_a)^{1/2} > (m_a + m_b)$$

$$\frac{\vec{v}}{c} = \frac{m\gamma v}{m\gamma c} = \frac{m_a \gamma_a v_a}{(m_a \gamma_a + m_b) c} \Rightarrow v = \frac{m_a \gamma_a}{m_a \gamma_a + m_b} v_a$$

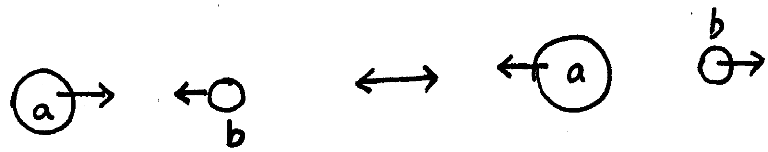
Center of mass frame (zero momentum frame).

Example 2: elastic head on collision

Lab frame S:



in the CM frame S':



In the lab frame, the finial state will be difficult to calculate. But in the CM frame, they just reverse the direction. (simple!). Let us find the relation between lab frame & CM frame.

· Lab frame: total momentum

$$P^{\mu}_{tot} = (\vec{P}_a, \frac{E_a}{c}) + (0, m_b c) = (\vec{P}_a, \frac{E_a + m_b c^2}{c})$$

in the C-M frame which has relative velocity βc respect to Lab frame

$$P'_1 = \gamma (P_a - \frac{\beta}{c} (E_a + \frac{m_b c^2}{c})) = 0 \Rightarrow \quad \beta = \frac{P_a c}{E_a + m_b c^2}$$

In this frame, the finial state of b can be obtained from

the initial frame $P_b = (0, m_b c)$ in the Lab frame

$$\left.\begin{aligned} P'_{b,in,1} &= \gamma (0 - \beta m_b c) = -\gamma m_b c \\ P'_{b,in,4} &= \gamma (-\beta \cdot 0 + m_b c) = \gamma m_b c \end{aligned}\right\} \Rightarrow \quad 1$$

$\Rightarrow$ in the CM frame. $P'^{b}_{final} = (\gamma\beta m_b c, \ \gamma m_b c)$.

back to Lab frame

$$P^{b}_{1,final} = \gamma(P'_{1,final} + \beta P_{4,final}) = \gamma[\gamma\beta + \gamma\beta] m_b c$$
$$= 2\gamma^2\beta m_b c$$

$$P^{b}_{4,final} = \gamma[\beta P'_{1,final} + P'_{4\,final}] = \gamma[\gamma\beta^2 + \gamma] m_b c$$
$$= \gamma^2[\beta^2+1] m_b c$$

$\Rightarrow$ the final velocity of B $\boxed{\frac{P_1 c}{P_4} = \frac{2\gamma^2\beta}{\gamma^2(\beta^2+1)} c = \frac{2\beta}{1+\beta^2} c}$

§ thereshold energies.

$$a + b \longrightarrow d + \cdots g$$

incident particle (a), target (b)

which is the minimum energy of a in the lab frame to creat $d, \cdots g$.

Lab frame $\quad P^{\mu}_{tot} = (P_a, \ E_a/c) + (0, \ m_b c) = (P_a, \ \frac{E_a + m_b c^2}{c})$

$\rightarrow$ ~~lab~~ frame Center of M $\quad \beta = \dfrac{P_a c}{E_a + m_b c^2}$

in this frame: ~~a particle~~ the total energy

$\cancel{cP_a, \frac{E_a}{c}}$ $$(E_{cm,tot}/c)^2 = \left(\frac{E_a + m_b c^2}{c}\right)^2 - P_a^2$$

$$\Rightarrow E_{cm}^2 = E_a^2 + m_b^2 c^4 + 2 E_a m_b c^2 - P_a^2 c^2 = (m_a^2 + m_b^2) c^4 + 2 E_a m_b c^2$$

$$E_{cm} \geq \sum m_{fin} c^2$$

$$\Rightarrow m_a^2 c^4 + m_b^2 c^4 + 2 E_a m_b c^2 \geq (\sum m_{fin})^2 c^4$$

$$\Rightarrow \boxed{E_a \geq \frac{(\sum m_{fin})^2 - m_a^2 - m_b^2}{2 m_b} c^2}$$

§ Massless particle - photon

$E = |P| C$, massless particle only travels with velocity c

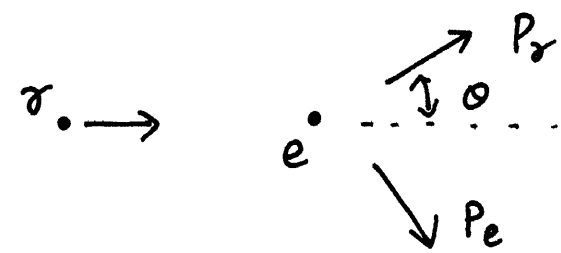
$$p^\mu p_\mu = p^2 - \frac{E^2}{c^2} = 0$$

wave-interpretation $\vec{P} = \hbar \vec{k}$, $E = \hbar\omega$, $\omega = kc$

$$p^\mu = \frac{\hbar\omega}{c}(\hat{k}, 1).$$

§ Compton scattering: light scattering from free electrons

inelastic



scattered light has longer wave length.

$$P^\mu_{in,\gamma} = \frac{\hbar\omega_0}{c}(\hat{k}_0, 1) \qquad P^\mu_{out,\gamma} = \frac{\hbar\omega}{c}(\hat{k}, 1)$$

$$P^\mu_{e,in} = (0,0,0,mc) \qquad P^\mu_{e,out} = ?$$

$$P^\mu_{in,\gamma} + P^\mu_{in,e} = P^\mu_{out,\gamma} + P^\mu_{e,out}$$

$$\Rightarrow P^\mu_{e,out} = P^\mu_{in,\gamma} + P^\mu_{int,e} - P^\mu_{out,\gamma}$$

$$P^\mu_{e,out}\cdot P_{\mu,e out} = P^\mu_{in,\gamma}\cdot P_{\mu,in\gamma} + P^\mu_{int,e}P_{\mu\, int e} + P^\mu_{out,\gamma}P_{\mu,out,\gamma}$$
$$+ 2P^\mu_{in\gamma}P_{\mu,in,e} - 2P^\mu_{in\gamma}P_{\mu\, out\gamma} - 2P^\mu_{in e}P^\mu_{out\gamma}$$

$$m_e^2c^2 = m_e^2c^2 + 0 + 0 + 2P^\mu_{in,e}(P_{in\gamma\mu} - P_{out,\gamma,\mu}) - P^\mu_{in,\gamma}P_{\mu,out,\gamma} {}^{=0}$$

$$P^\mu_{in,e}(P_{in,\gamma,\mu} - P_{out,\gamma,\mu}) = P^\mu_{in,\gamma}P_{\mu,out,\gamma}$$

only the 4th-compont is nonzero

$$mc\left[\frac{\hbar\omega_0}{c} - \frac{\hbar\omega}{c}\right] = \frac{\hbar^2\omega\omega_0}{c^2}[1 - \hat{k}\cdot\hat{k}_0]$$

$$\Rightarrow mc[\omega_0 - \omega] = \frac{\hbar}{c}\omega\omega_0(1-\cos\theta)$$

$$\Rightarrow \frac{1}{\omega} - \frac{1}{\omega_0} = \frac{\hbar}{mc^2}(1-\cos\theta) \Rightarrow \lambda = \frac{c}{\omega} \Rightarrow \Delta\lambda = \frac{\hbar}{mc}(1-\cos\theta)$$